PRACTICAL SCHOOL ALGEBRA

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PREFACE

The initial difficulties of Algebra are due to the novelty of the notation. They are best overcome, (i) by leading the pupil to think in numbers when using letters, which is done by regarding the subject as a generalisation of Arithmetic, (ii) by illustrating the practical utility of the notation in its application to formulae. These two ideas have governed the selection of the subject-matter of the first three chapters; as soon as the contents of these have been assimilated, all substantial difficulties of the subject disappear.

Algebra is a natural link between Arithmetic and Practical Geometry; illustrations should be drawn from each side to make the arguments intelligible and to break down the natural tendency to divide elementary mathematics into three water-tight compartments. Accordingly the fundamental facts of geometry are introduced into many of the examples and problems in this book. Free use has been made of diagrams, especially in the early stages, to illustrate arguments and to focus the attention of the pupil on the essential data of a problem. Some explanation is needed of the insertion of numerous "Supplementary Exercises" and "Extra Practice Exercises." The former are intended primarily for those pupils who tend to run ahead of the majority of the class, the latter for those who need additional "drill"; both sets of exercises will be found useful for revision. There are also a large number of test papers which are divided into two groups: the "Easy Revision Papers" are suited for ordinary work; the sets of "Harder Revision Papers" are added for those who have been able to tackle the Supplementary Exercises.

The book is divided into three Parts, which are issued separately as well as in a single volume. Each Part corresponds to a year's course. In order to facilitate the use of the book in its *separate* parts, the revision papers which appear at the beginning of Part II and of Part III are also bound up with Part I and Part II respectively. It is believed that this procedure will add materially to the usefulness of the book at schools where it is used in its separate-part form.

The author gratefully puts on record the extremely valuable help he has received from Mr. H. F. Atwill, B.Sc., who has criticised, page by page, the MSS. of this book, and has made numerous suggestions for its improvement. He is also indebted to Mrs. Palmer, for permission to make use of some of the material of *Elementary Algebra*, Part I. (Durell and Palmer); and finally he wishes to acknowledge a more general, but none the less real, obligation to Professor T. P. Nunn, whose books (*The Teaching of Algebra* and *Exercises in Algebra*: Longmans) cannot fail to influence the outlook of all teachers who know them.

C. V. D.

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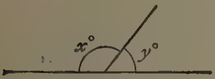
GEOMETRICAL THEOREMS

THE following formulae or theorems have been assumed at different stages of the book to secure greater variety of illustration and to link up Algebra and Geometry. They are summarised below for convenience of reference.

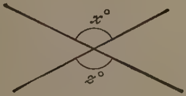
I. ANGLES.

(i)



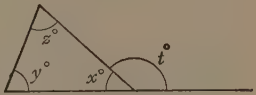
$x^\circ + y^\circ = 180^\circ.$

(ii)



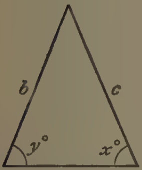
$x^\circ = z^\circ.$

(iii)



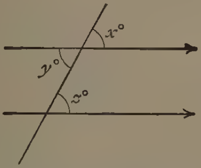
$t^\circ = y^\circ + z^\circ,$
$x^\circ + y^\circ + z^\circ = 180^\circ.$

(iv)



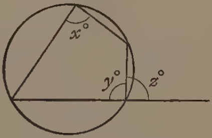
If $b = c$, $x^\circ = y^\circ$; and conversely.

(v)



For parallel lines $x^\circ = z^\circ = y^\circ.$

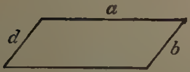
(vi)



For a cyclic quadrilateral.
$x^\circ + y^\circ = 180^\circ,$
$x^\circ = z^\circ.$

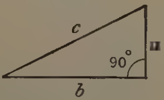
II. LENGTHS.

(i)



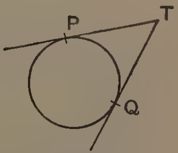
For a parallelogram (or rectangle).
$a = c$; $b = d.$

(ii)



$a^2 + b^2 = c^2.$

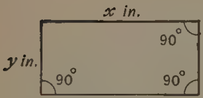
(iii)



$TP = TQ.$

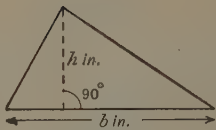
III. Areas.

(i)



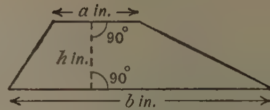
Rectangle.
Area $= xy$ sq. in.

(ii)



Triangle.
Area $= \frac{1}{2}hb$ sq. in.

(iii)



Trapezium.
Area $= \frac{1}{2}h(a+b)$ sq. in.

IV. Other Mensuration Formulae ($\pi \simeq \frac{22}{7} \simeq 3{\cdot}14$).

For a circle of radius r in.

Length of circumference	$= 2\pi r$ in.
Area of circle	$= \pi r^2$ sq. in.

For a circular cylinder, radius r in., height h in.

Area of curved surface	$= 2\pi rh$ sq. in.
Volume of cylinder	$= \pi r^2 h$ cu. in.

For a circular cone, base radius r in., height h in., slant edge l in.

Area of curved surface	$= \pi rl$ sq. in.
Volume of cone	$= \frac{1}{3}\pi r^2 h$ cu. in.

For a sphere, radius r in.

Area of surface	$= 4\pi r^2$ sq. in.
Volume of sphere	$= \frac{4}{3}\pi r^3$ cu. in.

PART I

CHAPTER I

USE OF LETTERS IN ALGEBRA

Generalisation.

THE use of letters in generalising statements should be explained orally and illustrated by simple examples.

Example I. Figure 1 represents a number of straight rods; part of each, shown as shaded, is painted black and the remainder of each is white. Find the length of the white portion of each rod and give a general statement which includes all these results.

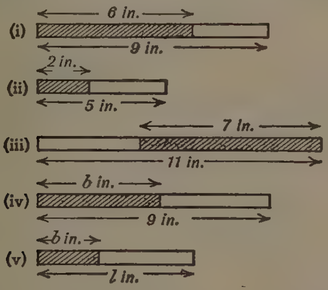


FIG. 1.

The first rod (i) is 9 in. long, and the black portion is 6 in. long; $\therefore$ the white portion is $(9-6)$ inches or 3 inches long.

Similarly, the lengths of the white portions of the rods in (ii) and (iii) are $(5-2)$ inches and $(11-7)$ inches.

In (iv), the black portion is b in. long; $\therefore$ the white portion is $(9-b)$ inches long.

Suppose in (i), the rod is l in. long and the black portion is 6 in. long, then the white portion is $(l-6)$ inches long.

We generalise these results as follows :

If the length of the rod is l inches (*see Fig.* 1 (v)), *and if the length of the black portion is b inches, then the length of the white portion is $(l-b)$ inches.*

Each of the previous results is a special case of the last statement. By using letters to represent numbers, it is possible to include in a single statement a number of different arithmetical facts. Algebra is therefore a mathematical form of shorthand.

Note. Letters are used to represent **numbers**. Do not use them to represent quantities, *i.e.* numbers of things. Do not say that the length of a line is l, but say that its length is l inches or l feet or l cm., etc.

Example II. Give a general statement to include the following statements :

$$2 \times 3 = 3 \times 2 ;\quad 5 \times 9 = 9 \times 5 ;\quad 7 \times 4 = 4 \times 7 ;\quad 6 \times 12 = 12 \times 6.$$

In each case, taking two numbers, we state that, when the first is multiplied by the second the result is the same as when the second is multiplied by the first.

Using letters to represent numbers, we say :

If x and y are any two numbers, then $x \times y = y \times x$.

Note. Numbers represented by letters **need not be whole** numbers. Suppose, for example, $x = 2\frac{1}{2}$ and $y = 3\frac{1}{5}$; then the general statement above becomes $2\frac{1}{2} \times 3\frac{1}{5} = 3\frac{1}{5} \times 2\frac{1}{2}$.

There is a danger that some pupils may imagine that a general statement has been *proved* to be true when it has merely been *verified* in a few special cases. This point should be emphasised orally, and it may be helpful to illustrate it by *discussing orally* one or more of the following :

(i) $1^2+1+41=43$ is prime, $2^2+2+41=47$ is prime, $3^2+3+41=53$ is prime, $4^2+4+41=61$ is prime, etc. Is x^2+x+41 prime, if x is a whole number ? [Obviously not, if $x=41$.]

(ii) In the same way, take the values of $2x^2+29$ for $x=1, 2, 3, \ldots$. [Obviously it is not prime if $x=29$.]

(iii) Take the sum of an even number of prime numbers from 1 upwards :

$$1+2, \qquad 1+2+3+5,$$
$$1+2+3+5+7+11, \qquad 1+2+3+5+7+11+13+17, \text{ etc.}$$

Each of these sums is a prime number. Can this result be generalised ?

Symbols.

The meanings of the symbols $=$, $>$, $<$ should be illustrated orally, if they are not already familiar.

$=$ means "is equal to" : thus $5-2=3$ and $4 \times 5 = 20$.

$>$ means "is greater than" : thus $5>2$ and $3\frac{1}{2} > 2\frac{3}{4}$.

$<$ means "is less than" : thus $2<5$ and $2\frac{3}{4} < 3\frac{1}{2}$.

There are two other useful symbols of the same kind :

$\simeq$ means "is approximately equal to" : thus $3\frac{1}{7} \simeq 3{\cdot}14$.

$\neq$ means "is not equal to" : thus $5 \neq 2$ and $2 \neq 5$.

Exercise I. *a*. (Oral.)

1. State in words the following :

(i) $\frac{10}{2}=5$; $7>4$; $5<8$; $2\frac{1}{2}<4<5\frac{1}{3}$; $10>7\frac{1}{4}>4$.

(ii) $N>3$; $A<10$; $x \neq 2$; $x=y=3$; $5>B>3$.

(iii) $\sqrt{2} \simeq 1{\cdot}41$; $x \neq y$; $\pi \simeq 3{\cdot}14$.

2. Write in symbols the following :

(i) 3^2 is greater than 3 ; $(\frac{1}{2})^2$ is less than $\frac{1}{2}$.

(ii) A is equal to $5\frac{1}{2}$; x is not equal to nought.

(iii) N is a number greater than 8.

(iv) B is a number less than 6.

(v) C is a number between 10 and 20.

(vi) Of the two numbers x and y, the greater is x.

(vii) An approximation for π is $\frac{22}{7}$.

Exercise I. *b*. (Oral.)

1. Figure 2 represents three glasses of different heights containing water to different depths. How much can the

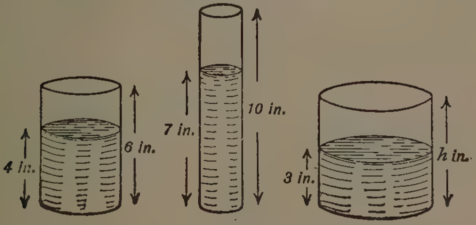


Fig. 2.

water level rise in each case before the water overflows? Answer the same question for a glass 5 in. high, containing water d in. deep and for a glass h in. high, with water d in. deep.

2. There are two parcels of unequal weights in the scale pans of a weighing machine. What weight must be placed in the

right-hand scale pan to make them balance, in the following cases ?

	(i)	(ii)	(iii)	(iv)	(v)
Weight in left pan -	6 lb.	10 lb.	8 lb.	W lb.	W lb. $W>w$
Weight in right pan	4 lb.	3 lb.	w lb.	$2\frac{1}{2}$ lb.	w lb.

Make a general statement.

3. (i) With the data of Fig. 3, state the distances of P and Q from B. What is the distance of R from B ?

(ii) How far is R from B, if $AB=s$ yd. and $AR=480$ yd. ?

(iii) How far is R from B, if $AB=s$ yd. and $AR=a$ yd. ?

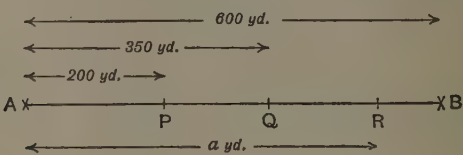


FIG. 3.

4. Part of a rod, see Fig. 4, is painted red, another part is white, and the rest is black.

	(i)	(ii)	(iii)	(iv)	(v)
Total length of rod in cm. -	10	12	18	l	l
Length of red part in cm. -	5	7	$3\frac{1}{2}$	4	r
Length of white part in cm.	3	4	$4\frac{1}{4}$	7	w

Find the length of the black part in each case. Also make a general statement.

FIG. 4.

5. Find the height of a pile of equal note-books in each of the following cases :

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
Number of note-books - - -	10	12	20	n	15	n
Thickness of each book, in cm. -	2	3	$1\frac{1}{2}$	2	t	t

6. Write down, without simplifying, the number of pence in 4s.; 13s.; 17s.; $11\frac{1}{2}$s. Give also a general statement.

7. Fig. 5 represents one straight line meeting another straight line. Write down the size of each unmarked angle and give a general statement.

FIG. 5.

8. Fig. 6 represents a rectangular shed $EDGF$ in the corner of a rectangular garden: the dimensions are given in feet.

Find the lengths of AE and CG, (i) if $b=10$, $l=15$, (ii) if $b=12$, $l=20$, (iii) in terms of b and l.

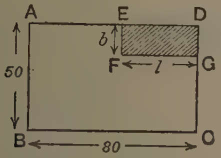


FIG. 6.

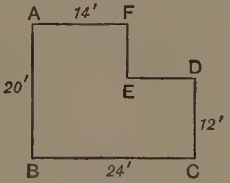


FIG. 7.

9. Fig. 7 represents the floor of a room with right-angled corners.

Find the lengths of DE, EF, (i) with the data in the figure, (ii) if $AB=b$ ft., $BC=l$ ft., $CD=12$ ft., $AF=14$ ft., (iii) if $AB=b$ ft., $BC=l$ ft., $CD=x$ ft., $AF=y$ ft., and draw a plan of the room if $b=15$, $l=20$, $x=5$, $y=10$.

10. Fig. 8 gives the dimensions in yards of three rectangular fields. Find the total length of fencing required for each

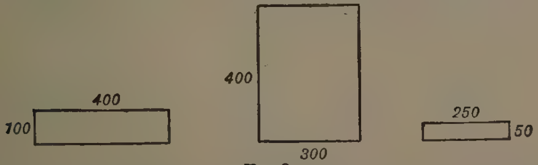


FIG. 8.

field. Give also a general statement for the perimeter of any rectangle, b yards long, w yards wide.

Give a general statement, by means of letters, to include all the statements in each of the examples 11-36.

11. $3+3=\text{twice } 3,$
$4+4=\text{twice } 4,$
$7+7=\text{twice } 7,$
$N+N=$

12. $2-2=0,$
$3-3=0,$
$8-8=0,$
$A-A=$

13. $3\times 3=3^2,$
$7\times 7=7^2,$
$13\times 13=13^2,$
$x\times x=$

14. $2+3=3+2,$
$5+8=8+5,$
$12+73=73+12,$
$P+Q=$

15. $\frac{1}{3}\times 3=1,$
$\frac{1}{7}\times 7=1,$
$\frac{1}{12}\times 12=1,$
$\frac{1}{N}\times N=$

16. $5\times\frac{1}{5}=1,$
$6\times\frac{1}{6}=1,$
$11\times\frac{1}{11}=1,$
$x\times\frac{1}{x}=$

17. $\frac{3\times 2}{5\times 2}=\frac{3}{5};\ \frac{3\times 7}{5\times 7}=\frac{3}{5};\ \frac{3\times 10}{5\times 10}=\frac{3}{5}.$

18. What must be added to 3 to obtain 11? Answer, $11-3$.
What must be added to 3 to obtain 19?
Answer, $19-3$.
What must be added to 3 to obtain 36?
Answer, $36-3$.
What must be added to 3 to obtain N?

19. 8 exceeds 5 by $8-5$.
10 exceeds 2 by $10-2$.
140 exceeds 27 by $140-27$.

20. By what must 7 be multiplied to obtain 21? Answer, $\frac{21}{7}$.
By what must 7 be multiplied to obtain 35?
Answer, $\frac{35}{7}$.
By what must 7 be multiplied to obtain 40?
Answer, $\frac{40}{7}$.
By what must 7 be multiplied to obtain 11·2?
Answer, $\frac{11\cdot 2}{7}$.
By what must 7 be multiplied to obtain n?

21. $7-1$, 7, $7+1$ are three consecutive whole numbers.
$12-1$, 12, $12+1$ are three consecutive whole numbers.
$39-1$, 39, $39+1$ are three consecutive whole numbers.

22. $2 \times 5 = 10$, and this is an *even* number.
$2 \times 13 = 26$, and this is an *even* number.
$2 \times 24 = 48$, and this is an *even* number.
If N is any whole number, $2 \times N$ is ....

23. $2 \times 5 + 1$ is an odd number.
$2 \times 7 + 1$ is an odd number.
$2 \times 12 + 1$ is an odd number.

24. $2 \times 6 - 1$ is an odd number.
$2 \times 9 - 1$ is an odd number.
$2 \times 20 - 1$ is an odd number.

25. $0 \times 4 = 0$.
$0 \times 7 = 0$.
$0 \times 18 = 0$.
$0 \times N =$

26. 3 times $5 + 5 = 4$ times 5.
3 times $6 + 6 = 4$ times 6.
3 times $11 + 11 = 4$ times 11.
3 times $t + t =$

27. Since $5 - 3 = 2$; $\therefore$ $5 = 2 + 3$.
Since $11 - 7 = 4$; $\therefore$ $11 = 4 + 7$.
Since $20 - 6 = 14$; $\therefore$ $20 = 14 + 6$.
If $a - b = c$, then ....

28. What must be subtracted from 20 to leave 4? Answer, $20 - 4$.
What must be subtracted from 20 to leave 7?
What must be subtracted from 20 to leave a, if $a < 20$?
What must be subtracted from b to leave a, if $b > a$?

29. The next even number above the even number 12 is $12 + 2$. What is the next even number above x, if x is an even number? What is the next even number above x, if x is an odd number?

30. How many whole numbers are there *between* (i) 6 and 10, (ii) 3 and 11, (iii) the whole numbers x and y, given that $y > x$?

31. Write down 5 consecutive whole numbers, such that the middle number is (i) 6, (ii) 10, (iii) the whole number x.

32. Up to and including 8, there are $\frac{8}{2}$ even numbers.
Up to and including 14, there are $\frac{14}{2}$ even numbers.

Up to and including 20, there are $\frac{20}{2}$ even numbers.
Up to and including N, where N is an even number, there are ....

33. Up to and including 7, there are $\frac{8}{2}$ odd numbers.
Up to and including 13, there are $\frac{14}{2}$ odd numbers.
Up to and including 19, there are $\frac{20}{2}$ odd numbers.

34. Below 6, there are $\frac{6}{2}$ odd numbers.
Below 12, there are $\frac{12}{2}$ odd numbers.
Below 18, there are $\frac{18}{2}$ odd numbers.

35. The sum of 4, 5, 6 is three times 5.
The sum of 9, 10, 11 is three times 10.
The sum of 56, 57, 58 is three times 57.
The sum of N, $N+1$, $N+2$ is ....
The sum of $n-1$, n, $n+1$ is ....

36. The sum of 5, 7 is twice 6.
The sum of 14, 16 is twice 15.
The sum of 37, 39 is twice 38.

37. Can you say at a glance the number of crosses in each of the groups, (i), (ii), (iii)? How many are there in the various compartments of each group?

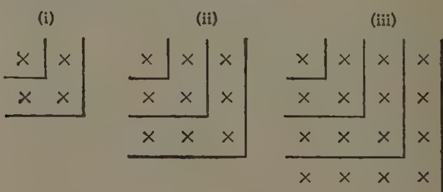


FIG. 9.

What is the value of (i) $1+3$, (ii) $1+3+5$, (iii) $1+3+5+7$?

Draw a group containing 5 rows of crosses with 5 crosses in each row and divide it up in the same way. What is the value of $1+3+5+7+9$?

What is the sum of (i) the first 6 odd numbers, (ii) the first 20 odd numbers, (iii) the first n odd numbers?

38. What is the sum of $1-1+1-1+1-1+\dots$, if you take (i) 10 terms, (ii) 15 terms, (iii) n terms ?

39. Take a number of groups of crosses as in Fig. 9 and add them up *along diagonals*. Prove in this way from group (iii) that $1+2+3+4+3+2+1=4^2$. Can you generalise this result ?

Notation.

The symbols $+$, $-$, $\times$, $\div$ have the same meaning in Algebra as in Arithmetic.

The product of 2 and 3 is represented by 2×3 or $2 . 3$; in the same way, the product of a and b may be represented by $a \times b$ or $a . b$; it is, however, generally represented by ab.

Similarly, $3x$ represents the product of 3 and x ; in a product, a number is always put *before* a letter, thus $3x$, not $x3$. In particular, just as $1 \times 7 = 7$, so $1 \times x = x$; the expression $1x$ is not used.

Note. (i) If $x=3$, $y=7$, $xy=3 \times 7=21$; xy is not equal to 37, because 37 means $3 \times 10+7$.

(ii) If $x=3$, $y=\frac{1}{2}$, $xy=3 \times \frac{1}{2}=\frac{3}{2}$; xy is not equal to $3\frac{1}{2}$, because $3\frac{1}{2}$ means $3+\frac{1}{2}$.

In other respects, the symbols used in Algebra have the same meaning as in Arithmetic.

7^4 means $7 \times 7 \times 7 \times 7$; a^4 means $a \times a \times a \times a$.

$\sqrt{8}$ means the square root of 8 ; $\sqrt{x}$ means the square root of x.

$\frac{3}{4}$ is the same as $3 \div 4$; $\dfrac{a}{b}$ is the same as $a \div b$.

$x \sim y$ means "the difference between x and y," *i.e.* the result of subtracting the smaller of the two numbers x and y from the larger.

Brackets.

The object of using brackets is to bind numbers together ; the contents therefore of a bracket may be regarded as equivalent to a single number.

Thus, $(7+3)$ means the number obtained by adding 3 to 7, and $(a+x)$ means the number obtained by adding x to a.

The product of 5 and $(7+3)$ is written $5(7+3)$; the product of 5 and $(a+x)$ is written $5(a+x)$.

Therefore $5(a+x)$ means "add x to a, multiply the result by 5."

Similarly, $(p-q) \div 7$ means "subtract q from p, divide the result by 7." This is often written in the form $\dfrac{p-q}{7}$.

Example III. What is the meaning of $2x+5$?

To obtain $2x+5$, we multiply 2 by x (or x by 2) and add 5 to the result.

Example IV. If x stands for 5, for what does $2x^3$ stand?

$2x^3$ stands for $2 \times x \times x \times x$,

i.e. for $2 \times 5 \times 5 \times 5$, if x stands for 5, *i.e.* for 250.

Note. $2x^3$ does not stand for $2x \times 2x \times 2x$. $2x \times 2x \times 2x$ could be represented by $(2x)^3$.

Example V. If a stands for 3 and b for 2, for what does a^2+5ab stand?

a^2 stands for $a \times a$, *i.e.* 3×3 or 9.

$5ab$ stands for $5 \times a \times b$, *i.e.* $5 \times 3 \times 2$ or 30.

$\therefore$ a^2+5ab stands for $9+30$ or for 39.

Note. $5ab$ does not stand for $5a \times 5b$. $5a \times 5b$ could be represented by $(5a)(5b)$ or $25ab$.

Example VI. If r stands for 10, what is the value of $\dfrac{r+2}{r}$?

If $r=10$, $\dfrac{r+2}{r}=\dfrac{10+2}{10}=\dfrac{12}{10}=\dfrac{6}{5}=1\frac{1}{5}$.

Example VII. What is the meaning of $6(c-d)$?

$(c-d)$ means the number obtained by subtracting d from c.

$\therefore$ to obtain $6(c-d)$, subtract d from c and multiply the result by 6.

Example VIII. If $p=9$, $q=16$, what are the values of $\sqrt{p}$, $\sqrt{q}$, $\sqrt{(p+q)}$?

$$\sqrt{p}=\sqrt{9}=3;\quad \sqrt{q}=\sqrt{16}=4.$$

To obtain $\sqrt{(p+q)}$, add q to p and take the square root of the result.

Here, $$p+q=9+16=25;$$
$$\therefore\ \sqrt{(p+q)}=\sqrt{25}=5.$$

Note. $\sqrt{p}+\sqrt{q}=3+4=7$. This is not the same as $\sqrt{(p+q)}$.

Exercise I. c. (Oral.)

1. Give the meanings of the following (see Examples III and VII):

(i) $2a$; (ii) $3b+c$; (iii) x^3; (iv) p^3+p;

(v) $2hk$; (vi) $4ac^2$; (vii) $xy-y$; (viii) $\dfrac{x}{3}$;

(ix) $\dfrac{12}{z}$; (x) $\dfrac{ab}{a}$; (xi) $\dfrac{r+s}{r}$; (xii) $\sqrt{(2x)}$;

(xiii) $\sqrt{x}+2$; (xiv) y^2-y; (xv) $bc-bd$; (xvi) rst.

2. How do you write more shortly (i) $7 \times 7 \times 7 \times 7 \times 7$; (ii) $x \times x \times x \times x \times x \times x$? How would you write it if there were eleven factors, each of which is x?

3. In what other ways can you write (i) $2 \times b^2$, (ii) $y^2 \times y^3$?

4. If x stands for 10, for what do the following stand?

(i) $2x$; (ii) $2 \div x$; (iii) $x+2$; (iv) x^2;

(v) $3x^2$; (vi) $\frac{x}{2}$; (vii) $x+3x$; (viii) $2x-2$;

(ix) $10-x$; (x) $\frac{2}{x}$; (xi) $\sqrt{(x-1)}$; (xii) $(3x)^2$?

5. If a stands for 2 and b stands for 3, for what do the following stand?

(i) $a+b$; (ii) $b+a$; (iii) ab; (iv) ba;

(v) $b-a$; (vi) b^2-a^2; (vii) $3a-2b$; (viii) $5a+2b$;

(ix) $10ab$; (x) a^2b; (xi) ab^2; (xii) $\frac{b}{a}$;

(xiii) $\frac{b+5}{a}$; (xiv) b^2-b; (xv) $3ab^2$.

6. If $x=3$ and $y=2x$, what is the value of y?

7. If $x+y=10$ and $y=2$, what is the value of x?

8. If $pr=48$ and $r=8$, what is the value of p?

9. If $s=3t$ and $t=2$, what is the value of st?

10. State in words *how to obtain* the values of the following expressions; do *not* simplify them:

(i) $3(a-b)$; (ii) $3a-b$; (iii) $2(y+z)$;

(iv) $2y+z$; (v) $p+3(r+s)$; (vi) $c-(d+e)$;

(vii) $c \div (d-e)$; (viii) $(x+y)-(a+b)$; (ix) $c(d-e)$;

(x) $3a(y-z)$; (xi) $(3+a)(y+z)$; (xii) $(a-t)(b-t)$.

EXERCISE I. *d.* (Written.)

1. If $x=5$ and $y=2$, find the values of the following:

(i) $x-2y$; (ii) $2xy$; (iii) y^3; (iv) xy^2;

(v) $xy-x$; (vi) $\frac{x^2}{y}$; (vii) $\frac{2x}{y}$;

(viii) x^2-y^3; (ix) $3y^2$; (x) $\sqrt{(8y)}$.

2. If $a=\frac{1}{2}$, find the values of the following:

(i) $1-a$; (ii) a^2; (iii) $2a$; (iv) $a+2$;

(v) $\frac{1}{a}$; (vi) $\frac{a}{3}$; (vii) $2a^3$; (viii) $6a-3$.

3. If $r=2$, $s=5$, $t=3$, find the values of the following:

(i) rst; (ii) $3rst$; (iii) $r+s+t$; (iv) $5(r+s)$;

(v) $s-r-t$; (vi) $r(s-t)$; (vii) rst^2; (viii) $\frac{st}{r}$.

4. If $c=3$, find the values of the following:

(i) c^2-1; (ii) $2c^3$; (iii) $\frac{c}{12}$; (iv) $\frac{24}{c}$;

(v) c^2-3c; (vi) $3c^2-5c+2$; (vii) $\frac{c+1}{2c}$; (viii) $c(c+2)$;

(ix) $\frac{c+1}{c-1}$; (x) $(c+1)(c+2)$; (xi) $(c-1)^2$; (xii) $(2c)^2$.

5. If $p=0$, $q=2$, find the values of the following:

(i) $p+1$; (ii) $7p$; (iii) $p+q$; (iv) $2pq$;

(v) p^2q; (vi) p^2+q^2; (vii) $3qp+q^2$; (viii) $\frac{p}{q}$.

6. If $R=\frac{1}{3}$, $r=\frac{1}{5}$, find the values of the following:

(i) $R-r$; (ii) $2Rr$; (iii) $10Rr$; (iv) $\frac{R}{r}$;

(v) $R+1$; (vi) $\frac{1}{r}$; (vii) $\frac{2}{R}$; (viii) $\frac{4}{R}-\frac{2}{r}$.

7. If $y=2x^2+x$ and $x=3$, what is the value of y?

8. If $p+2q=r$, what is the value of r when $p=1$ and $q=3$?

9. If a is the square of b, and if $b=3c$ and if $c=2$, what is the value of abc?

10. If $u-v=3$ and if $v=5$, what is the value of $u+v$?

11. If $\frac{x}{y}=4$ and if $y=3$, what is the value of xy?

12. If $x=2y=3z=24$, what is the value of $x+y+z$?

EXTRA PRACTICE EXERCISES. E.P. 1.

SUBSTITUTION.

1. If $a=3$, $b=5$, what are the values of

(i) ab; (ii) $a+b$; (iii) $2a-b$; (iv) ba^2;

(v) $2a^2$; (vi) $\frac{2a}{b}$; (vii) $b-a$; (viii) $1+ba$?

2. If $u=3$, $s=1$, $t=0$, what are the values of

(i) us; (ii) us^2; (iii) uts; (iv) $u^2+s^2+t^2$;
(v) $2u-3s$; (vi) $us+st$; (vii) $3t$; (viii) $4s^3-3ut$?

3. If $x=6$, $y=0$, $z=3$, what are the values of

(i) $x+y+z$; (ii) $2x-3y-2z$; (iii) $xy+yz$;

(iv) $\frac{x}{z}$; (v) $\frac{1}{z}-\frac{1}{x}$; (vi) xz^2;

(vii) $xz-2xy$; (viii) $\frac{yz}{x}$?

4. If $N=4$, $n=\frac{1}{2}$, what are the values of

(i) Nn; (ii) $\frac{1}{n}$; (iii) Nn^2; (iv) $N-5n$;

(v) $\frac{N}{n}$; (vi) $\frac{n}{N}$; (vii) $2n^2$; (viii) $\frac{1}{2}N^2$?

5. If $p=4$, $q=9$, $r=0$, what are the values of

(i) $\sqrt{p}$; (ii) $\sqrt{q}$; (iii) $\sqrt{(pq)}$; (iv) pqr;

(v) q^2-pr; (vi) $6p-2q+3r$; (vii) $\frac{q-1}{p}$; (viii) $\frac{r}{q}$?

6. If $m=1$, $n=2$, $t=3$, what are the values of

(i) m^3; (ii) mnt; (iii) $m+n-t$; (iv) $t^2-m^2-n^2$;

(v) n^2-mt; (vi) n^3; (vii) $\frac{m+n}{t}$; (viii) $\frac{nt}{m}$?

7. If $e=3$, $f=5$, $g=0$, what are the values of

(i) $4ef$; (ii) $2f^2$; (iii) $3fg+ge$; (iv) $5e^2-6f$;

(v) $3e^2-f^2$; (vi) fge^2; (vii) e^3+2g^2; (viii) $\frac{g}{ef}$?

8. If $a = b^2 - b$ and $b = 5$, what is $\frac{a}{b}$?

9. If $x = 2y$ and $y = 3z$ and $z = 4$, what is x?

10. If $2m^2 + n^2 = p$ and if $m = 0$ and $n = 1$, what is p?

11. If $x = 2\frac{1}{2}$ and $y = 1\frac{2}{3}$, what is $\frac{1}{x} + \frac{1}{y}$?

12. If $u = \frac{2}{v}$ and $v = \frac{3}{4}$, what is (i) uv, (ii) $\frac{u}{v}$?

Generalisations.

Make general statements for the following:

13. The cost of 7 lb. of sugar at 3 pence per lb. is 3×7 pence.

14. In £2 there are 2×8 half-crowns.

15. The number of $1\frac{1}{2}$d. stamps sold for 12 pence is $\frac{12}{1\frac{1}{2}}$.

16. From 10 a.m. to 5 p.m. is $(12 - 10) + 5$ hours.

17. If a man earns £500 a year and spends £40 a month, he saves £$(500 - 40 \times 12)$ a year.

18. If a boy scores 55 runs in 5 completed innings, his average is $\frac{55}{5}$ runs.

19. If a boy bicycles at 9 miles an hour, he travels 6 miles in $\frac{6}{9}$ hours.

20. A is the same amount older than B as B is older than C. If C is 9 years old and B is 13 years old, then A is $13 + (13 - 9)$ years old.

21. If an empty coal-scuttle weighs 3 lb. and the scuttle, when full, weighs 14 lb., the coal in it weighs $(14 - 3)$ lb.

22. If a railway fare is 18 pence, 4 whole tickets and 5 half-tickets cost $(18 \times 4 + 18 \times \frac{5}{2})$ pence.

23. If a soldier's stride is 30 inches, he takes every hundred yards $\frac{100 \times 36}{30}$ paces.

24. If a clock loses 5 seconds each hour, it loses $\frac{5 \times 24}{60}$ minutes each day.

25. $\frac{3}{3}=1$; $\frac{6}{6}=1$; $\frac{11}{11}=1$.

26. $7\times 0=0$; $4\times 0=0$; $10\times 0=0$.

27. $5\times 5\times 5=5^3$; $8\times 8\times 8=8^3$; $12\times 12\times 12=12^3$.

28. $1+\frac{2}{3}=\frac{3+2}{3}$; $1+\frac{7}{11}=\frac{11+7}{11}$; $1+\frac{5}{14}=\frac{14+5}{14}$.

29. $0\div 3=0$; $\frac{0}{12}=0$; $0\div 10=0$.

30. $2+4+6=3\times 4$; $7+9+11=3\times 9$; $14+16+18=3\times 16$.

Numbers and Quantities.

Letters in Algebra are used to represent *numbers*, **not** numbers-of-things. A letter may stand for 2, 15, $\frac{3}{4}$, etc., but **not** for 2 pence, 15 days, $\frac{3}{4}$ mile. A number-of-things is called a **quantity**. When dealing with quantities, always state what the unit is; thus 3 lb., W lb., £5, £$(m+2)$, $2\frac{1}{2}$ ft., $(p-q)$ feet, $\frac{a+b}{2n}$ yards, $\frac{x}{y}$ pints, etc.

Exercise I. *e*.

1. I walk s miles and then ride 4 miles. How many miles have I travelled?

2. A box weighs 3 lb.; I put into it two parcels, one weighing W lb., the other w lb. What is the total weight?

3. A milkman sells m gallons of milk out of 10 gall.; how much has he left?

4. A boy starts with d pence and then spends n pence; how much has he left?

5. In a bookcase, there are $2x$ books on one shelf, 20 books on the next shelf and $2y$ books on the remaining shelf. How many books are there altogether?

6. I buy n apples and then buy 2 more; how many do I buy in all?

7. I buy a bunch of $3t$ bananas and cut off 3 of them; how many are left?

8. A basket, weighing W lb. when empty, contains n lb. of apples; if c lb. of apples are now sold, what is the weight of the basket with the remaining apples?

9. It is now a quarter past ten; in how many minutes' time will it be (i) 10.30, (ii) t minutes past ten, (iii) n minutes to 11?

10. I can walk 4 miles an hour. How far can I walk in 3 hrs.; $2\frac{1}{2}$ hrs.; b hrs.; $5b$ hrs.?

11. What is the cost of the following:

(i) 5 lb. of butter at l pence per lb.
(ii) 2 oz. of pepper at x pence per oz.
(iii) m yd. of silk at 12 shillings per yd.
(iv) x gall. of oil at p shillings per gall.
(v) $2R$ ft. of pipe at n pence per ft.
(vi) N dozen books at P pence each book.

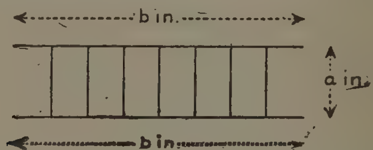
12. How many inches are there in 4 ft., 4 ft. 9 in., l ft., l ft. 9 in., l ft. m in., p ft. q in., v yd. z in.?

13. I spend 5 shillings out of £2. How many shillings are left?

I spend 5 shillings out of £N. How many shillings are left?

I spend f shillings out of £g. How many shillings are left?

14. What is the total length (in inches) of the wire used to make the grid shown here?



Each inch of wire weighs $\frac{1}{5}$ oz.; what is the weight of the grid in ounces?

What is the value of each answer if $a=2$, $b=8$?

15. Eggs are 3 pence each. How much change (in pence) is there out of half-a-crown, if you buy z eggs?

16. A railway porter is paid b shillings a week and receives in tips n shillings a day. He works 6 days a week, how much does he get each week?

17. A car uses a gallon of petrol every 24 miles. (i) How far will the car run on $2\frac{1}{2}$ gall., $\frac{3}{4}$ gall., t gall.? (ii) How much petrol is used to run 1 mile, 10 miles, s miles?

18. A man buys a horse for £40 and sells it for £P; what is his profit?

19. By selling a cricket bat for z shillings I gain 10 shillings, how much did it cost me?

20. (i) Share v apples among 5 boys. How many does each have? (ii) Share v apples among 5 boys and x girls. How many does each have?

21. A jug holds 2 pints? How many jugs are needed for k pints, n gallons?

22. A pail holds g pints. How many times can it be filled from a tank containing 100 pints, n pints, z gallons?

23. Express in yards, 16 ft., 2 ft., l ft.

24. Express in ounces, 2 lb., $1\frac{1}{2}$ lb., W lb.

25. A family uses 3 loaves of bread a day. How many loaves are needed for n days, t weeks? How long will x loaves last?

26. How long (in hours) does it take to travel 120 miles at 30 m.p.h., v m.p.h.? How long to travel s miles at u m.p.h.?

27. A match-box weighs a gm. and the matches in it weigh b gm. What is the weight of a full box of matches in gm.? What is the weight of a packet of 1 dozen boxes of matches, if the paper wrapper weighs w gm.?

28. In a garden-city, there are N houses per acre. How many houses are there per sq. mile? [1 sq. mile = 640 acres.] How many acres are needed for 2000 houses?

29. A boy is x years old and his father is three times as old. How old is his father? How old will his father be in 3 years' time?

30. The houses in a row are numbered from 1 to N; each house has 3 windows in front. How many windows are there in front altogether?

31. A row of houses is l yards long; each house is w yards wide. How many houses are there in the row?

32. A bookshelf is a feet long; how many books, each t inches thick, will it hold?

33. The price of coal is 2s. per cwt. on Sept. 1st; it increases by p pence per cwt. during the next 2 months. What is the price in pence per cwt. on Nov. 1st?

34. A boy goes to bed at 9 p.m. and gets up at x a.m.; how many hours is he in bed?

35. A boy goes to bed at 10 p.m. and stays in bed t hours $(2 < t < 12)$, what time does he get up?

36. If I cycle at v miles per hour, it takes me n hours to get from my house to the station; how far away is the station?

37. Take the number n; square it and multiply the result by 6.

38. Take the number p; multiply it by 4 and subtract 4.

39. Take the number x; divide it by 2 and multiply it by 3.

40. A gramophone costs £P; n boys each contribute 2 shillings, but this is not enough. How many more shillings are required?

41. A ship starts with W tons of coal as fuel and uses n tons per day. How many tons are left after t days?

42. The shadow cast by a telegraph pole on the ground is l feet long when the shadow of a man 6 feet high is 3 feet long. How high is the pole?

43. A clerk's salary is £100 for the first year and increases by £p each year. How much is it for the third year?

44. What length of wire is obtained from n coils of wire, each containing x yards? Answer (i) in yards, (ii) in miles.

45. What is the total length of wire obtained from m coils of wire each containing y yards and p coils of wire each containing z yards? Answer (i) in yards, (ii) in miles.

46. A man buys 100 cigarettes and smokes n cigarettes a day; how many are left after a week? How long will the hundred last him?

47. A man walks s miles from his house to a town at 4 miles per hour; how long does he take? He stays there 2 hours and walks back again at v miles an hour. How long is he away from his house altogether?

48. Two clocks are put right at noon on Sunday; one gains g seconds per day, the other loses l seconds per day. What is the difference between them one week later?

49. At an entertainment, N people buy one-shilling tickets and n people buy sixpenny tickets. How much money is taken (i) in pence, (ii) in shillings?

50. There are two exit-doors from a hall. People pass out through one door at the rate of x per minute and through the other at the rate of y per minute. How many come out in 5 minutes? How long does it take for N people to come out?

51. A man buys R pairs of boots at 15 shillings per pair and sells them at B shillings per pair. What is his profit?

52. A man buys n eggs at 2d. each; 12 are broken, the rest are sold at 3d. each. What is his profit?

53. The sum of the ages of n children is p years. What will be the sum of their ages in 2 years' time?

54. A man pays 1d. for his daily paper and 2d. for his Sunday paper. How much (in shillings) does he pay for n weeks? For how long will $2P$ shillings pay for his newspapers, P being a whole number?

55. A bath has cold-water and hot-water taps which admit 10 gallons per min. and 8 gallons per min. respectively. The waste-pipe allows 6 gallons per min. to run away. The bath holds 180 gallons.

(i) If both taps are turned on, how many gallons flow in per min.? How long will the bath take to fill?

(ii) If the hot-water tap is turned on and the waste-pipe is open, how long will it be before the bath overflows?

(iii) If both taps are turned on and the waste-pipe is open, how long will it take to fill the bath?

56. Repeat No. 55, assuming that the cold tap admits c gall. per min., the hot tap h gall. per min., while the waste-pipe allows w gall. per min. to run away.

57. Each coach in a train is l feet long and the engine is e feet long. The total length of the train is s yards. How many coaches are there in the train?

58. A crate of iron plates is being loaded up for a crane. The plates are all 5 ft. by 2 ft. and are made of sheet metal weighing b lb. per sq. foot. What is the weight of one plate? The crate weighs W lb. and there are 6 plates in it. What is the total weight lifted by the crane?

59. If, in No. 58, the chain of the crane can bear a load of N cwt. without breaking, what is the largest number of plates that can safely be placed in the crate?

60. A spiral spring is suspended from one end. Its unstretched length is a cm., and the extension is k cm. for each gram weight of load attached to the other end. What load causes the spring to stretch to a length of b cm.?

Use of Brackets.

Example IX. 8 oz. of cocoa are packed in a tin which weighs t oz. when empty; 30 tins of cocoa are packed in a box weighing W oz. What is the total weight?

One tin, when full of cocoa, weighs $(8+t)$ oz.

$\therefore$ 30 tins of cocoa weigh $30(8+t)$ oz.

$\therefore$ the case of cocoa weighs $W+30(8+t)$ oz.

Exercise I. *f*.

Use brackets when answering the following questions; do **not** *remove the brackets.*

1. 10 lb. of jam is put in a jar which weighs w lb. when empty. What is the weight of 10 full jars? The jars are packed in a box weighing P lb. What is the weight of the box, when packed?

2. A cellar contains n bottles; six dozen of them hold a pint each and the rest a quart each. How many pints are there altogether?

3. A man buys glasses at the rate of 6d. each for the first dozen, and 5d. each for every additional glass. What is the cost in pence (i) of 20 glasses, (ii) of x glasses where $x>12$?

4. A man at a hotel is charged £1 a day for the first four days, and 16s. a day afterwards. What is the amount of his bill in shillings, (i) for 7 days, (ii) for n days, if $n>4$?

5. A journey by car takes $3\frac{1}{2}$ hours; for the first t hours, the speed is 20 miles per hour, and for the rest of the time it is 15 miles per hour. What is the distance travelled? Assume $t<3\frac{1}{2}$.

6. A man's rate of pay is as follows: ordinary time, 10d. an hour; overtime 15d. an hour. The ordinary working day is 7 hours. How much does he receive for a day on which he works, (i) 5 hours, (ii) 10 hours, (iii) x hours if $x<7$, (iv) x hours, if $x>7$?

7. The letter rate for inland postage is as follows: $1\frac{1}{2}$d. for the first 2 oz. and $\frac{1}{2}$d. for each additional 2 oz. or part thereof. What is the cost in pence of sending a letter weighing W oz., if W is an even whole number?

8. What is the answer for No. 7, if W is an odd whole number?

Write down expressions for the following in Nos. 9-19; *do not remove the brackets.*

9. The result of subtracting $x+y$ from $2z$.

10. The result of subtracting $x-y$ from $a-b$.

11. Twice the remainder after p is subtracted from q.

12. The product of $2l$ and $l-m$.

13. Half the sum of a, b, c.

14. The quotient when $x+y$ is divided by $c+d$.

15. The square of the sum of x and y.

16. The sum of the squares of $R+r$ and $R-r$.

17. The excess of 1 over $s+t$.

18. The product of three consecutive integers of which (i) n is the least, (ii) n is the middle integer.

19. The number of pence in $(a+b)$ shillings.

20. A man walks 34 miles in 3 days; he walks $(x+y)$ miles in the first 2 days; how far does he walk on the third day?

21. From a stick l in. long, a part $(x+y)$ in. long is cut off. What is the length of what is left?

22. A book costs $\left(x-\frac{x}{4}\right)$ shillings, and is sold for 15 shillings. What is the profit?

23. When a boy is x years old, his father is y years old. How much older is that? How old is the boy when his father is z years old?

24. A rectangular field is $(c+d)$ yards long and its area is 10 acres or 48,400 sq. yards. What is the breadth of the field?

25. A box full of sugar weighs W lb. and when empty weighs w lb. How much sugar is there in n boxes?

26. A cask when empty weighs P lb. and when full weighs Q lb. What is the weight of the cask when half full?

27. A man is earning $(a+3b)$ shillings a week. What are his new wages if this is increased by one-tenth?

28. A photograph, see Fig. 10, is mounted on a sheet of cardboard l in. long, h in. high; there is a margin t in. wide all

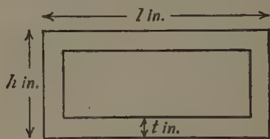


FIG. 10.

the way round. What is (i) the length of the photograph, (ii) its height, (iii) its area? What is the area of the margin?

Construction of Formulae.

Example X. The measurement of Horse-Power.

An engine, hoisting 1 lb. through 1 ft., does 1 ft.-lb. of work.

An engine, hoisting 550 lb. through 1 ft., does 550 ft.-lb. of work.

An engine, hoisting 550 lb. through 300 ft., does 550×300 ft.-lb of work.

If the engine does this work in 2 minutes it does

$$\frac{550 \times 300}{2} \text{ ft.-lb. per minute.}$$

But an engine that does 33,000 ft.-lb. per minute is called a 1 Horse-Power engine and is said to have a power of 1 H.P.

$$\therefore \text{ this engine's horse-power is } \frac{550 \times 300}{2 \times 33,000} \text{ H.P.}$$
$$= 2\tfrac{1}{2} \text{ H.P.}$$

Now every time an engineer wishes to find the H.P. of an engine which is capable of hoisting a given weight through a given height

in a given time he would have to work through the above Arithmetic unless he could invent a shorter method.

Generalising the problem, let us say

W lb. stands for the weight to be hoisted,

h ft. stands for the height to which it is to be raised,

t min. stands for the time to be taken.

If we then repeat the argument used above, we seo that the power of the engine must be $\frac{W \cdot h}{33{,}000t}$ horse-power.

If then the engine is of H horse-power, we see that

$$H = \frac{W \cdot h}{33{,}000t},$$

whatever the weight, height and time may be.

Therefore the value of H can be calculated rapidly for any given conditions.

The relation just obtained is called a formula, and an engineer would put this into his note-book for future use.

Example XI. The Area of a Rectangle.

If you have a rectangle, 4 in. long, 3 in. broad, you can divide it, as in Fig. 11, so that there are 3 rows and each row contains 4 one-inch squares. Therefore there are 4×3 one-inch squares altogether. But the area of a one-inch square is called 1 sq. inch.

$\therefore$ the area of the rectangle is 4×3 sq. in.

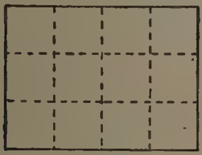


FIG. 11.

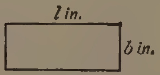


FIG. 12.

Now it would be very tedious to go through this process every time we wished to find the area of a rectangle. *We therefore look for the general formula.*

Suppose a rectangle is l inches long and b inches broad. By repeating the argument used above, we see that its area is $l \times b$ sq. inches.

Call the area A sq. inches.

Then $\mathbf{A = l \cdot b}$ is the required formula.

Note. Before constructing the general statement or formula, invent a numerical example of the same kind for yourself. Work this out and

then use the same method when working with letters. Whenever you do this, state clearly what is given and what is required in the example you have invented.

Exercise I. *g*.

[If brackets occur in the Answer, do *not* remove them.]

Find the necessary formulae in Nos. 1-10.

1. The cost in pence of an inland telegram (n words).

2. The third angle of a triangle, given the other two angles ($A°$, $B°$).

3. The time taken by a train, running at v miles an hour, to travel a given distance (s miles).

4. When making tea, put in one spoonful for each person and one for the pot. How much tea is required for n people, if (i) one teapot is used, (ii) if k teapots are required?

5. The cost of n collars, priced as follows:

Number bought -	1	4	12
Total cost - -	1s. 3d.	5s.	15s.

6. The cost of string, sold by weight as follows:

Weight in oz. -	8 oz.	12 oz.	3 lb.
Price - - -	1s. 4d.	2s.	8s.

7. Using No. 6, find the weight of a ball of string which costs x shillings y pence.

8. The speed of the tips of the sails of a windmill and the velocity of the wind are connected as follows:

Speed of tips in ft./min. -	V	1040	2080	2600
Velocity of wind in ft./min.	v	400	800	1000

What do you notice about the value of $\frac{V}{v}$ in the given cases? What is the probable formula connecting V and v?

9. The prices of various sizes of vases of a certain design are shown in a trade catalogue as follows :

Size	1	2	3	4
Height in inches	**6** or 3×2	**9** or 3×3	**12** or 3×4	**15** or 3×5
Price in shillings	**2** or $6 - 4$	**5** or $9 - 4$	**8** or $12 - 4$	**11** or $15 - 4$

The catalogue shows only those numbers put in thick type ; the others are added to help you.

(i) What is the probable height of size 5 and of size N ?
(ii) What is the probable cost of size 5 and of size N ?

10. Use the table in No. 9 to answer the following :

(i) A vase costs 20 shillings, what is its height and size ?
(ii) A vase costs P shillings, what is its height and size ?
(iii) A vase is 30 inches high, what is its price and size ?
(iv) A vase is h inches high, what is its price and size ?

11. A member of a club receives three free tickets, and is charged 5s. for each additional ticket. How much does he pay for (i) 7 tickets, (ii) p tickets, (a) in s., (b) in £ ?

12. A book is x in. thick, each cover is y in. thick and there are n sheets. What is the thickness of each sheet ?

13. A piece of cotton is wound n times round a cylinder. When unwound, it measures l inches. What is the length of the circumference of the cylinder, (i) in inches, (ii) in feet ?

14. The wheel of a car makes r revolutions when the car travels s yards. What is the circumference of the wheel, (i) in yd., (ii) in ft. ?

15. What is (i) the perimeter, (ii) the area of the rectangle in Fig. 13 ?

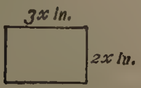


FIG. 13.

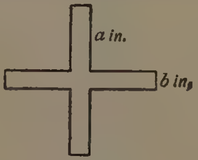


FIG. 14.

16. Fig. 14 represents a cross with four equal arms. What is (i) the perimeter, (ii) the area of the cross ?

17. How many times can a jug holding p pints be filled from a cask holding g gallons? [1 gallon = 8 pints.]

18. In Fig. 15, AB is longer than BC by the same amount that BC is longer than CA. What is the distance of A from B?

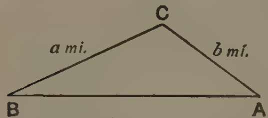


FIG. 15.

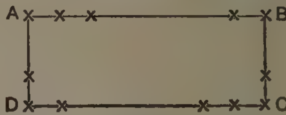


FIG. 16.

19. How many posts are required for a straight fence AB, $5l$ yards long (see Fig. 16), if a post is needed every 5 yards; l being an integer?

20. How many posts are required for the fence of a field $ABCD$ (see Fig. 16), $5l$ yards long, $5b$ yards broad, if a post is required every 5 yards; l, b being integers?

21. A closed box, see Fig. 17, is x in. long, x in. wide, y in. deep, external measurements; find (i) the total length of all its edges, (ii) the total area of its outside surface, (iii) its volume.

What do these results become in terms of x, if $y = 2x$?

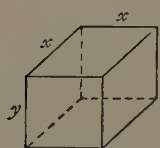


FIG. 17.

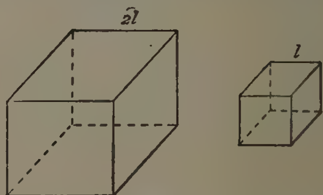


FIG. 18.

22. Fig. 18 represents two cubical tins, one of side $2l$ inches, the other of side l inches; the larger is full of water, the smaller is empty. How much water is left in the larger tin, when the smaller has been filled from it? How many more tins, each the same size as the smaller, can be filled from the larger? [Neglect the thickness of the tin.]

23. Neither tin in Fig. 18 has a lid. What is the area of the total surface of each tin?

24. Find the area of Fig. 19 by expressing it as the *sum* of two rectangles.

25. Find the area of Fig. 20 by expressing it as the *difference* of two rectangles.

26. Find the area of Fig. 21 by expressing it as the *difference* of two rectangles.

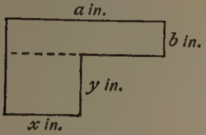


FIG. 19.

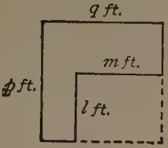


FIG. 20.

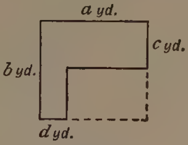


FIG. 21.

27. Find the perimeter of the areas bounded by the continuous lines in (i) Fig. 19, (ii) Fig. 20, (iii) Fig. 21.

28. In Fig. 22, $AB = b$ in., $AC = c$ in., and P is the mid-point of BC. What is the length of (i) PC, (ii) AP?

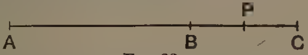


FIG. 22.

29. Fig. 23 represents a rectangular brick wall pierced by three equal windows, each h ft. high, w ft. wide. The units for

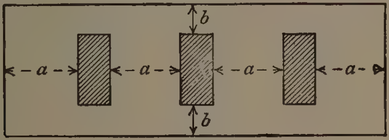


FIG. 23.

the dimensions shown in the figure are feet. What is (i) the length and height of the wall, (ii) the area of the brickwork?

30. The units for the dimensions shown in Fig. 24 are feet. What is the area of the figure?

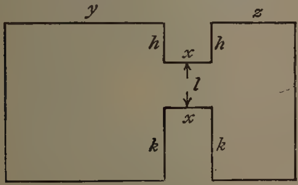


FIG. 24.

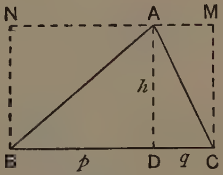


FIG. 25.

31. In Fig. 25, AD is the altitude of $\triangle ABC$; the units of the given dimensions are inches. What is the area of (i) $\triangle ADB$, (ii) $\triangle ABC$?

32. Fig. 26 represents a sheet of paper from which equal squares, side h in., have been cut away at each corner. The paper is folded to form a box, without a lid; the dotted lines show the creases. (i) Find the length, breadth, and height of the box. (ii) What is the volume of the box? (iii) Write down *without any working* the total area of its surface.

What do the results in (i), (ii), (iii) become in terms of h, if $a=5h$ and $b=4h$?

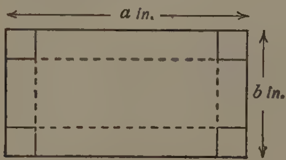


FIG. 26.

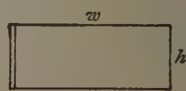


FIG. 27.

33. Look at an ordinary (Bryant and May's) match-box: it consists of a drawer which slides through a cover with two open ends. The open ends are w in. wide, h in. high, and the box is l in. long. Find the area of matchwood used to make the complete box. Notice that the cover is constructed with an overlap at one side, as shown in Fig. 27, to secure it.

SUPPLEMENTARY EXERCISE. S. 1.

1. What angle in Fig. 28 is equal to:

(i) $[180-(x+y)]$ degrees,

(ii) $(x+y)$ degrees,

(iii) $[(x+y)-z]$ degrees?

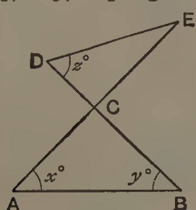


FIG. 28.

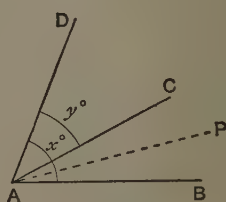


FIG. 29.

2. In Fig. 29, AP bisects $\angle BAC$.
Express in degrees (i) $\angle BAC$, (ii) $\angle BAP$, (iii) $\angle PAD$.

3. If in Fig. 30, $\angle ABC$ is a right-angle, Pythagoras' theorem states that $a^2+c^2=b^2$.

If $AC=(x+y)$ in. and $BC=(x-y)$ in., express, without simplifying, the length of AB in terms of x, y.

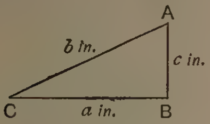


FIG. 30.

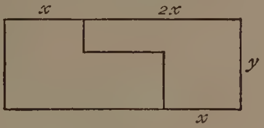


FIG. 31.

4. In Fig. 31, all the angles are right-angles, and the units for the given dimensions are inches. Find the sum of the lengths of all the lines in the figure.

5. The Moon was 8 days old on July 18, 1926. Write down a formula for the age of the Moon (x days) on the nth of July, 1926. What is the least value of n for which your formula can be used ?

6. The driving wheel A (Fig. 32) has N teeth and is making R revolutions per minute ; the driven wheel B has n teeth. If

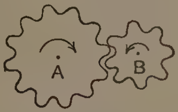


FIG. 32.

B is making r revs. per min., find the formula for r in terms of N, R, n.

7. A train, travelling on the level at v miles per hour, can be pulled up by its brakes within x yards, where v, x are connected as follows :

v	30	36	45	54	60
x	100	144	225	324	400

What is the value of $\dfrac{v^2}{x}$ in each case ? Suggest a probable formula connecting v and x.

8. The lamp B in Fig. 33 is raised x inches by pulling A,

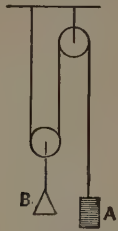


FIG. 33.

the counterpoise, y inches down. Construct a formula connecting x and y.

9. An ordinary horse, pulling a vehicle at v miles an hour, exerts a tractive force of P lb. weight where P, v are connected as follows :

v	2	3	4	5
P	165	110	$82\frac{1}{2}$	66

Evaluate $v \times P$ in each case. Suggest a formula expressing P in terms of v.

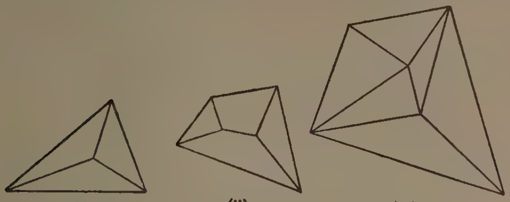
10. A double-action lift and force pump of 1 H.P. can raise per hour G gallons of water from a well h ft. deep, where G, h are connected as follows :

h	880	550	360	250	220
G	225	360	550	792	900

Can you find a formula for G in terms of h ? A man, by turning a crank, can raise 225 gallons per hour from a well 80 ft. deep. What would you expect him to raise per hour from a well, (i) 50 ft. deep, (ii) 100 ft. deep ?

11. Draw another figure formed of straight lines, like Fig. 34 (i), (ii) and (iii) ; number it (iv). Count up the number of

vertices V, the number of compartments C and the number of sides S, for each figure and tabulate the results.



(i) (ii) (iii)

FIG. 34.

	V	C	S	Value of $V+C$
(i) - -				
(ii) - -				
(iii) - -				
(iv) - -				

Can you find a formula which expresses S in terms of V, C ?

12. Take an ordinary closed box and count up the number of its faces F, the number of its corners C, and the number of edges E. Repeat this for other solids bounded by plane faces, *e.g.* a triangular prism, a triangular pyramid, a solid L, and one other solid you select yourself. Tabulate the results.

	F	C	E	Value of $F+C$
Ordinary box -				
Triangular prism -				
Triangular pyramid				
Solid L - - -				
Another solid -				

Can you find a formula which expresses E in terms of F, C ?

Use of Formulae.

Example XII. From a masthead h feet above the surface of the sea it is possible to see a distance of $\sqrt{\left(\frac{3h}{2}\right)}$ miles. How far is it possible for an observer to see if he is (i) at the top of a mast 54 feet above the sea, (ii) on the top of a cliff 150 feet high?

(i) Put $h=54$.

The distance of the horizon is $\sqrt{\left(\frac{3\times 54}{2}\right)}$ miles

$$=\sqrt{(81)}=9 \text{ miles.}$$

(ii) Put $h=150$.

The distance of the horizon is $\sqrt{\left(\frac{3\times 150}{2}\right)}$ miles

$$=\sqrt{(225)}=15 \text{ miles.}$$

Example XIII. Give two numerical examples of the following statement: If x is a whole number, $x+1$ is a factor of x^2-1.

(i) Let x stand for 6.

The statement becomes $6+1$ is a factor of 6^2-1
or 7 is a factor of 35.

(ii) Let x stand for 10.

The statement becomes $10+1$ is a factor of 10^2-1
or 11 is a factor of 99.

Each of these special statements is obviously true.

The process of deducing special results from a general formula is called **substituting in the formula.**

Exercise I. *h*.

1. n postcards cost $(n+1)$ pence if $n<12$, and cost $(n+2)$ pence if $11<n<23$. What is the cost of (i) 8 postcards, (ii) 20 postcards?

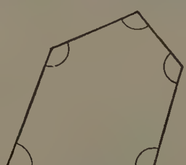


Fig. 35.

2. Fig. 35 represents a polygon with 5 sides (*i.e.* a pentagon). If a polygon has n sides, the sum of its interior angles is $(2n-4)$

right angles. What is the sum of the angles of (i) a quadrilateral, (ii) a pentagon, (iii) an octagon (8 sides)? Does the formula hold for a triangle?

3. After the xth day of November, there are $(61-x)$ days to the end of the year. How many days are there after November 25?

4. M miles may be taken as $\frac{8M}{5}$ kilometres. Express 100 miles in kilometres.

5. Assuming that the middle of the day is at 12 o'clock, if the sun rises at t o'clock, it sets at $(12-t)$ o'clock. On April 19, it rises at 5 o'clock; when does it set?

6. A parcel whose weight does not exceed $(3n-1)$ lb. may be sent by post for $(3n+3)$ pence; but parcels for which $n>4$ are not accepted. Interpret this rule numerically, assuming that n is an integer.

7. For corrugated iron roofing (see Fig. 36) the relation between the pitch, P in., and the depth, d in., is $d=\frac{1}{4}P$. The pitch should not be less than 3 inches; what is the least depth? The pitch should not be more than 5 inches; what is the greatest depth?

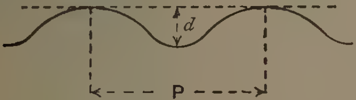


FIG. 36.

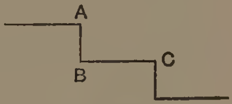


FIG. 37.

8. The "rise" AB, R inches, and the "tread" BC, T inches, of a staircase, see Fig. 37, are often connected by the rule, $R=\frac{1}{2}(24-T)$. The tread should not be less than 9 inches; what is the greatest rise? The tread should not be more than 12 inches; what is the least rise?

9. Repeat No. 8, using the rule, $R=\frac{66}{T}$, which is sometimes employed.

10. x miles an hour is the same speed as $\frac{22x}{15}$ feet per second. Express in ft. per sec. (i) 60 m.p.h.; (ii) 45 m.p.h.; (iii) 5 m.p.h.

11. If one of the acute angles of a right-angled triangle is $x°$, then the other one is $(90-x)$ degrees. Give two numerical examples of this statement.

12. $F°$ Fahrenheit is the same temperature as $C°$ Centigrade if $F = 32 + \frac{9C}{5}$ (see Fig. 38).

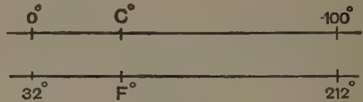


FIG. 38.

Express in degrees Fahrenheit (i) 100° C., (ii) 0° C., (iii) 15° C., (iv) 35° C.

[*Note.* 100° C. means "100 degrees Centigrade."]

13. The relation between F and C in No. 12 may also be stated in either of the following ways:

(i) $F = \frac{9}{5}(C + 40) - 40$; (ii) $C = \frac{5}{9}(F + 40) - 40$.

Show that these formulae agree for (i) 15° C., (ii) 50° F.

State *in words* how to convert degrees Fahrenheit to degrees Centigrade and *vice versa*.

14. $x + 1$ is a square root of $x^2 + 2x + 1$. Show that this is true when $x = 6$ and when $x = 0$.

15. If r and s are two consecutive whole numbers, r being the smaller, then $s^2 - r^2$ is equal to $s + r$. Show that this is true when $r = 8$ and when $r = 11$.

Deduce two arithmetical results from the statements in Nos. 16-18.

16. The product of c and $c + 1$ is $c^2 + c$.

17. If the square of $r - 1$ is subtracted from the square of $r + 1$, the result is $4r$.

18. If p is less than q and if both are whole numbers, there are $q - 1 - p$ whole numbers *between* p and q.

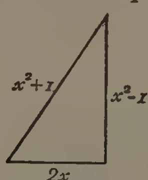


FIG. 39.

19. If the lengths of the sides of a triangle (see Fig. 39) are

x^2+1, x^2-1, $2x$ inches, then the triangle is right-angled. What results are obtained by taking (i) $x=2$, (ii) $x=3$, (iii) $x=4$?

20. One man A *after* n years' service is paid £$(200+15n)$ a year ; another man B *after* n years' service is paid £$(150+20n)$ a year ; how much does each receive in his 3rd, 10th, 12th year of service ? What did each receive for the first year ?

21. If the diameter of a circle is d feet, its area may be taken to be $\frac{11d^2}{14}$ sq. feet. What is the area of the circular top of a table of diameter 7 feet ?

22. A wind blowing at v miles an hour exerts a direct pressure of P lb. per sq. foot, where $P=\frac{v^2}{200}$. What pressure must a hoarding 10 ft. high, 20 ft. wide be able to withstand against (i) a breeze of 10 m.p.h., (ii) a gale of 25 m.p.h., (iii) a storm of 50 m.p.h. ?

23. The height (h feet) of a place above sea-level may be found by observing the temperature ($T°$ Fahrenheit) at which water boils and using the formula $h=520(212-T)+(212-T)^2$. What is the height of a place where the temperature of boiling water is 200° F. ?

24. An oak beam, l feet long, b inches wide, d inches thick, is built into a wall at one end (see Fig. 40), and carries a load

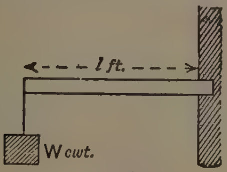


FIG. 40.

of W cwt. at the other end. It will break if $W>\frac{5bd^2}{4l}$. Will such a beam, 5 ft. long, 8 in. wide, 3 in. thick break under a load of 1 ton ?

SUPPLEMENTARY EXERCISE. S. 2.

1. An engineers' note-book states that the number (n) of years' life of an iron bar submerged in water is given by the formula $n = k \cdot \frac{W}{G}$, where W is the weight in lb. per foot-length, G is the girth in feet, and k is a constant depending on the nature of the water.

	Foul sea-water.	Clear sea-water.	Foul river-water.	Clear river-water.
k .	5	8	7	80

Find the life of a bar for which $G=2$ and $W=30$, in the four cases enumerated.

2. For a hawser rope, C inches in circumference, the breaking load, B tons, and the working load, W tons, are given by the formula $B = b \cdot C^2$ and $W = w \cdot C^2$, where b, w are constants depending on the composition of the rope.

	Russian hemp.	White manilla.	Best hemp.
b .	0·2	0·4	0·6
w .	0·04	0·06	0·1

Tabulate the breaking load and working load for the qualities of rope enumerated, for girths of 1, 2, 4 inches.

3. Under a head of water, H feet, the velocity of flow is v feet per second, where $v = 8\sqrt{H}$. What is v if $H = 25$?

4. For road repairs, stones are often heaped at the side along the road; Fig. 41 shows the cross-section of the heap. If

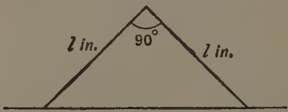


FIG. 41.

B bushels are required per yard length of road, $l = 12\frac{1}{4}\sqrt{B}$. What is l, if $B = 16$?

5. When a railway cutting, h feet high, is faced with a brick wall, the thickness of the wall at its base is t inches, where

$$t=\frac{h}{3}+3 \text{ if } h<18 \quad \text{and} \quad t=\frac{2h}{3}-3 \text{ if } h>18.$$

Find the thickness of the wall at the base if the height is (i) 15 ft., (ii) 21 ft.

How would you proceed for a height of exactly 18 feet ?

6. For a windmill of sail-area A sq. ft., when the wind is blowing at v ft. per sec., the H.P. developed is H, where

$$H=\frac{9Av^3}{10^7}.$$

For a sail-area of 60 sq. ft., find the H.P. in a wind blowing at 10 ft. per sec.

What sail-area is needed to develop 1 H.P. in a wind blowing at 20 ft. per sec. ?

7. When the track of a railway is curved, the outer rail is raised h inches above the inner rail, according to the formula

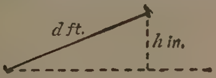


FIG. 42.

$h=\dfrac{12dV^2}{5R}$, where d ft. is the distance between the rails, R yards is the radius of the curve and V miles per hour is the maximum speed allowed.

Find h, if $d=4\frac{1}{2}$, $V=20$, $R=1800$ and state in words exactly what the result means.

8. The sum of the first n integers 1, 2, 3, 4, 5, ... is $\frac{1}{2}n(n+1)$. Show that this is true, when (i) $n=6$, (ii) $n=9$.

9. If n of the numbers 1, 4, 7, 10, 13, 16, 19, ... are written down, the last number is $3n-2$. Show that this is true when (i) $n=4$, (ii) $n=8$.

10. A set of numbers is written down so that the nth number in the set is n^2+n. What are the first five numbers in the set ?

11. If a circle touches the sides of a triangle, as in Fig. 43, it can be proved that $AQ = AR = \frac{1}{2}(AB + AC - BC)$.

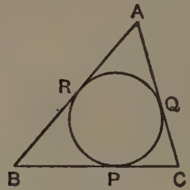


FIG. 43.

Find AQ if $AB = 8$ in., $BC = 7$ in., $CA = 5$ in.

Can you write down a similar formula for the length of BP or BR?

What is the length of BP in the special case given above?

12. From the formula $s = ut + \frac{1}{2}at^2$, find t when $s = 4$, $u = 2$, $a = 0$.

CHAPTER II

GENERALISED ARITHMETIC. PROCESSES. ADDITION AND SUBTRACTION

Like Terms.

Example I. Generalise the statement $9+9+9+9+9=9\times 5$.

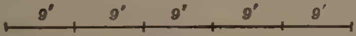


FIG. 44.

Suppose a straight fence (see Fig. 44) is made of 5 equal hurdles, each 9 ft. long; its total length $=9+9+9+9+9$ ft. $=9\times 5$ ft.

Suppose each hurdle is x ft. long.

Its total length $=x+x+x+x+x$ ft. $=x\times 5$ ft. $=5x$ ft.

$$\text{Thus}\quad \mathbf{x+x+x+x+x=5x.}$$

Example II. Generalise the statement $3\times 9+5\times 9=8\times 9$.

Suppose the fence in Example I. is made by first placing 3 hurdles each 9 ft. long and then adding on 5 hurdles each 9 ft. long.

The total length is $(3\times 9+5\times 9)$ ft.

But there are in all $3+5=8$ hurdles; $\therefore$ the total length $=8\times 9$ ft.

$$\therefore \text{ the total length} =(3\times 9+5\times 9) \text{ ft.} =8\times 9 \text{ ft.}$$

Suppose each hurdle is x ft. long.

The length of the first 3 hurdles $=3x$ ft., the length of the last 5 hurdles $=5x$ ft.; $\therefore$ the total length $=(3x+5x)$ ft.

But there are in all 8 hurdles; $\therefore$ the total length $=8x$ ft.

$$\therefore\ \mathbf{3x+5x=8x.}$$

Note. This is simply a *short-hand* statement.

$$3x=x+x+x \quad\text{and}\quad 5x=x+x+x+x+x;$$

$$\therefore\ 3x+5x=x+x+x+x+x+x+x+x=8x.$$

Example III. What is the value of $6x - x$?

FIG. 45.

Suppose we have a fence consisting of 6 hurdles each x ft. long: its total length is $6x$ ft.

Now take away the last hurdle (see Fig. 45), the reduced length of the fence is therefore $(6x - x)$ ft.

But there are now only 5 hurdles ; $\therefore$ the length is $5x$ ft.

$$\therefore\ 6x - x = 5x.$$

Note. This is simply a *short-hand* statement.

$$6x = x + x + x + x + x + x; \quad \therefore\ 6x - x = x + x + x + x + x + x - x$$
$$= x + x + x + x + x = 5x.$$

EXERCISE II. *a*. (Oral.)

Write down, *without* any calculation, short-hand forms for the following expressions:

1. $5 + 5 + 5 + 5$. 2. $a + a + a + a$. 3. $b + b + b$.
4. $x^2 + x^2$. 5. $xy + xy + xy$. 6. $7 + 7 + 7 + 7 + 7 + 7$.
7. $z + z + z + z + z + z + z + z + z$.
8. $x + x + x + x + \ldots$ to thirty terms.
9. $7 \times 11 + 8 \times 11$. 10. $7x + 8x$. 11. $5a + 5a$.
12. $5 \times 7 + 7$. 13. $5y + y$. 14. $3b^2 + 4b^2$.
15. $6 \times 9 - 2 \times 9$. 16. $6x - 2x$. 17. $10p - p$.
18. $ab + 2ab$. 19. $7y^2 - 2y^2$. 20. $3bc - bc$.
21. $2x^2 - x^2$. 22. $6ac - 3ac$. 23. $abc + abc$.
24. $x + 3x - 4x$. 25. $3a + 2a + a$.

Give concrete illustrations (cf. Example I. above) of the following equalities:

26. $a + a + a = 3a$. 27. $3l + 4l = 7l$.
28. $5R - 2R = 3R$. 29. $2xy + 3xy = 5xy$ [Areas].
30. $6c + 3c - 4c - 5c = 0$. 31. $11z - 7z + 2z = 6z$.

Unlike Terms.

Example IV. A farmer owns 5 cows and 8 goats. Each cow is worth £c and each goat is worth £g. What is the value of the stock?

The 5 cows are worth £$5c$. The 8 goats are worth £$8g$.

$\therefore$ the total value is £$5c$ + £$8g$ or £$(5c+8g)$.

Note. This cannot be put more simply unless we know the relation between the value of a cow and the value of a goat.

$5c$ and $8g$ are called **unlike terms.**

Example V. A man rides x miles in a bus and then walks y miles. How far does he go altogether?

The total distance is x miles $+ y$ miles

or $(x+y)$ miles.

Note. The result of adding y to x cannot be expressed more shortly, if x and y represent any numbers whatever. We say that x and y are *unlike terms.*

Example VI. A tourist walks at v miles an hour for 4 hours on the first day and at v miles an hour for t hours on the second day. How far does he walk in the two days?

The first day he walks $4v$ miles.

The second day he walks tv miles.

$\therefore$ the total distance is $(4v+tv)$ miles.

Note. $4v$ and tv are *unlike terms.*

We could of course say that he walked in the two days for $(4+t)$ hours at v miles an hour, and therefore he travelled $(4+t)v$ miles.

Here, we say that 4 and t are unlike terms, because we cannot express $4+t$ in any simpler way, unless we know the numerical value of t.

The expressions $4v+tv$ amd $(4+t)v$ are equal, and each involves the sum of two unlike terms.

Example VII. Find the short-hand form of

$$3a+2+a+8b-2a-3b.$$

In this expression $3a+a-2a$ are like terms,

also $8b-3b$ are like terms.

$3a+a-2a=2a$ and $8b-3b=5b$.

$\therefore$ the expression $=2a+5b+2$.

There is no shorter way of writing it, if a *and* b *represent any numbers whatever.*

Exercise II. *b*.

1. What is the length of a fence, formed by 10 hurdles each x ft. long and 12 hurdles each y ft. long?

2. What does it cost altogether to buy x lb. of tea at 2s. per lb. and y lb. of sugar at 4d. per lb.? Answer (i) in pence, (ii) in shillings.

3. Each mesh in Fig. 46 is l in. long and b in. broad. What is the total length of wire required for the network ?

FIG. 46.

4. A boy works for x hours on Sunday, y hours on each of Wednesday and Saturday, and z hours on each other day. How many hours does he work per week ?

5. From a rod $(3x+2y)$ inches long, a portion $2x$ inches long is cut off ; what length remains ?

6. Subtract $7x$ pence from x shillings. Answer in pence.

7. A man starts with £$(x+y)$. He pays x bills of 5 shillings each and y bills of 15 shillings each. How many shillings has he left ?

8. Equal holes, each x in. long, y in. wide, are punched in the metal sheet shown in Fig. 47. What is the surface-area of the sheet ?

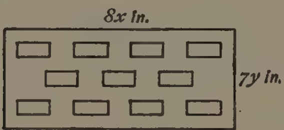


FIG. 47.

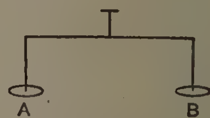


FIG. 48.

9. Fig. 48 represents a weighing machine with a body A weighing $(6x+3)$ lb. and a body B weighing $2x$ lb. in opposite scale pans. What weight must be used to make them balance ?

10. A rectangular garden is represented by the shaded area in Fig. 49 ; it is enlarged as shown in the figure, the units of the

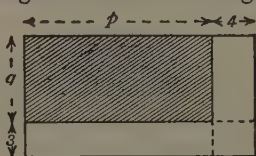


FIG. 49.

given dimensions being yards. What is (i) the final area, (ii) the increase of area ?

Write down (when possible) short-hand forms for the expressions in Nos. 11-37. If there is no shorter form, say so.

11. $a+a+b$.
12. $x+y+x+y+x$.
13. $b+2+b$.
14. $l+m+m+l+m$.
15. $p+3+p+5+p$.
16. $r+s+5+s+1$.
17. $b+b+c-b$.
18. $R+r-R+r$.
19. $z+2+z+z+1$.
20. $3R+r+2R$.
21. $8d-d-8$.
22. $3x+y-2x+y$.
23. $y+z-y$.
24. $3a+2b-a-b$.
25. $3p+5q-q-2p$.
26. $a+b+c+a+b+c$.
27. $x+y+z+x-y-z$.
28. $4a+b+2c-2a-b$.
29. $e+f+5+f+1$.
30. $2l+3m+4-l-2m-3$.
31. $3ab+3ac$.
32. $2xy+yz-xy$.
33. $3bc+3cb$.
34. $xy+x+y$.
35. $2xy+3x-yx$.
36. $5yz+2zy-yz-6yx$.
37. $ab+bc+abc$.
38. Add ab to ba.
39. Add xy to x.
40. Subtract b from bc.
41. Subtract yz from zy.
42. Add xy to xz.
43. Increase p by 1.
44. Decrease $2q$ by 2.
45. Add $R+r$ to $3R$.
46. Simplify $11 \times 23 - 10 \times 23$.
47. Simplify $6 \times 17 + 4 \times 17$.
48. Subtract 9×19 from 9×29.
49. Add 13×7 to 13×3.

50. In Fig. 50, $\angle ABC = (x+2y)$ degrees, $\angle BAC = (y+2x)$ degrees. What is $\angle ACD$?

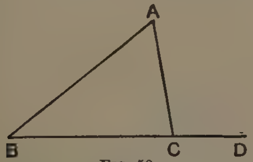


FIG. 50.

51. In Fig. 50, $\angle ACD = (3x+4y)$ degrees, $\angle ABC = (2x+y)$ degrees. What is $\angle BAC$?

Powers.

The short-hand form of $x \times x \times x \times x$ is $\mathbf{x^4}$ and the short-hand form of $7 \times x \times x \times x \times x$ is $\mathbf{7x^4}$.

The *numerical* factor 7 in the term $7x^4$ is called the **coefficient** of the term $7x^4$ or more shortly the **coefficient of $\mathbf{x^4}$**.

The term $7x^4$ is said to be of **degree 4 in x** or of the **4th degree in x**. The symbol x^4 is read as "x to the power 4" and the 4 is called the **index** of x.

Any group of terms, *e.g.* $2x^5 + 4x^3 + x^2 + 3$ is called an **expression**. In this expression, the coefficient of x^5 is 2, the coefficient of x^3 is 4, the coefficient of x^2 is 1, the coefficient of x is 0, since this term is missing. The *numerical* term 3 is called the **constant term** or the **term independent of x**, because its value does not depend on the value of x, since it does not contain x.

Example VIII. A shed A, $3x$ ft. long, $2x$ ft. wide, is built in a rectangular courtyard, leaving a passage 5 ft. wide along one side

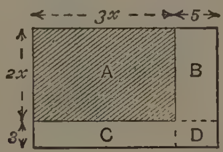


FIG. 51.

and 3 ft. wide along the other. What was the original area of the courtyard ?

The courtyard can be divided into four rectangles A, B, C, D, as shown in Fig. 51.

The area of A is $2x \times 3x = 2 \times 3 \times x \times x = 6x^2$ sq. ft.
The area of B is $5 \times 2x = 10x$ sq. ft.
The area of C is $3 \times 3x = 9x$ sq. ft.
The area of D is $3 \times 5 = 15$ sq. ft.

$\therefore$ the total area is $6x^2 + 10x + 9x + 15$ sq. ft.

Now $10x$ and $9x$ are like terms ; $\therefore\ 10x + 9x = 19x$.

$\therefore$ the total area is $\mathbf{6x^2 + 19x + 15}$ **sq. ft.**

Note. This expression cannot be written more shortly, unless we know the value of x, because $6x^2$ and $19x$ are *not* like terms.

$6x^2$ means $6 \times x \times x$ or $6x \times x$.

$19x$ means $19 \times x$.

Just as $7x + 3x = (7 + 3)x$, so $6x^2 + 19x = 6x \times x + 19 \times x = (6x + 19) \times x$, but this is no shorter, and still contains *two unlike terms*, viz. $6x$ and 19.

Expressions should always be written in an orderly way. They are usually arranged *either* in **descending powers** of the unknown,

i.e. beginning with the highest power, then the next highest, and so on, *or* in ascending powers, *i.e.* beginning with the constant term, then the term of first degree, then the term of second degree, and so on.

Thus $2x^5 - 4x^3 + x^2 + 3$ is arranged in *descending* powers of x.
And $3 + x^2 - 4x^3 + 2x^5$ is arranged in *ascending* powers of x.

Example IX. Simplify $2y^2 + 3y + 2 - y^2 + 2y + 1$ and arrange it in (i) descending powers of y, (ii) ascending powers of y.

Take the like terms together.

$$2y^2 - y^2 = y^2\ ;\ \ 3y + 2y = 5y\ ;\ \ 2 + 1 = 3.$$

$$\therefore \text{ the expression } = y^2 + 5y + 3, \text{ descending powers}$$
$$= 3 + 5y + y^2, \text{ ascending powers.}$$

Exercise II. *c*.

Find the area of the figures in Nos. 1-9, all the corners being right-angled, the units of the given dimensions being inches.

1.

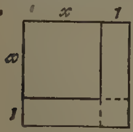


Fig. 52.

2.

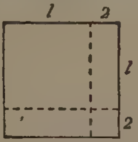


Fig. 53.

3.

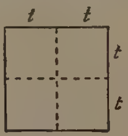


Fig. 54.

4.

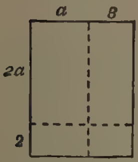


Fig. 55.

5.

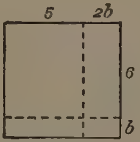


Fig. 56.

6.

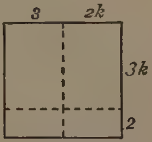


Fig. 57.

7.

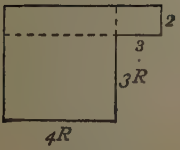


Fig. 58.

8.

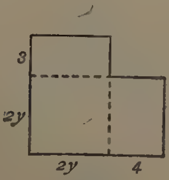


Fig. 59.

9.

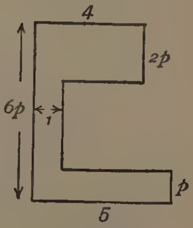


Fig. 60.

Find the volumes of the solids in Nos. 10-15, all the corners being right-angled, the units of the given dimensions being inches.

10. Fig. 52 represents the top of a box, x inches high.

11. Fig. 53 represents the top of a box, 3 inches high.

12. Fig. 54 represents the side of a box, t inches thick.

13. Fig. 55 represents the bottom of a box, $2a$ inches high.

14. Fig. 56 represents the bottom of a box, $10b$ inches high.

15. Fig. 57 represents the bottom of a box, $2k$ inches high.

Simplify, where possible, the following, and arrange the expressions in descending powers:

16. $6x^2-4x^2$.

17. $x^2+3x-2x$.

18. $a^3+3a^2+3a-a^2$.

19. $y^2-2y+1+1+2y+y^2$.

20. $t^2-1+t-1$.

21. a^4+2a^4.

22. $5b^2+8b-4b^2-8$.

23. $c^2+c+2+2-c$.

24. $y^4+9y^2+1-5y^2$.

25. $l^3+l^2+l+1+l+l^2$.

26. $5+6x+3x^2-5x$.

27. $2b+3b^2-1-b^2$.

28. $4a^2-ab-b^2+3ab$.

29. $3+5x^2-2x-1+x^2$.

30. $3r^2+2r^5-3r-2$.

31. $5s^3+10s+10s^2+5-5s^2-5s$.

32. $6+2p^3+4p^2-5-3p-p^3$.

33. The number 347 may be written $3 \,.\, 10^2+4 \,.\, 10+7$. In a similar way, write 265. What is the value of $2x^2+6x+5$, (i) if $x=10$, (ii) if $x=1$? Is there a short-hand form of $2x^2+6x+5$?

34. The number 2083 may be written $2 \,.\, 10^3+8 \,.\, 10+3$. In a similar way, write 4609. What is the value of $4x^3+6x^2+9$ (i) if $x=10$, (ii) if $x=1$? Is there a short-hand form of $4x^3+6x^2+9$?

35. Is there a short-hand form of

(i) x^2+x; (ii) x^2-x; (iii) $2x^3-x^3$; (iv) x^3-3?

36. What is the value of t^2+2t+9, (i) when $t=10$, (ii) when $t=1$? Is there a short-hand form of t^2+2t+9?

37. Write down (i) the term of degree 4, (ii) the coefficient of x^3, (iii) the constant term, in

(*a*) $3x^4+2x^3+4x+8$;

(*b*) $2x^5+4x^4+x^3+2x^2+9$;

(*c*) $2x^5+10$;

(*d*) $x^8+3x+x^2+5x^3$.

38. Write down (i) the term of highest degree, (ii) the coefficient of a, (iii) the constant term in

(1) $12a^3+2a^4+7a+2\frac{1}{2}$; (2) $a+100a^2+3a^3$.

39. Simplify and arrange (i) in descending powers, (ii) in ascending powers :

(*a*) $2+7t^2+t^3+9t^4+2t$; (*b*) $t+t^8+3t^2+1+4t^6$;

(*c*) $t^3+2t^2+3t+4-t-2t^2$;

(*d*) $t^3+1+3t^2+3t+1-3t+3t^2-t^3$.

Multiplication and Division.

Example X. To prove that $x^3\times x^2=x^5$.

$$x^3=x\times x\times x \quad\text{and}\quad x^2=x\times x.$$
$$\therefore\ x^3\times x^2=x\times x\times x\times x\times x=x^5.$$

Example XI. Multiply $3a^2$ by $2ab$.

$$3a^2=3\times a^2 \quad\text{and}\quad 2ab=2\times a\times b.$$
$$\therefore\ 3a^2\times 2ab=3\times a^2\times 2\times a\times b=3\times 2\times a^2\times a\times b$$
$$=\mathbf{6a^3b}.$$

Example XII. Divide x^6 by x^2.

$$x^6\div x^2=\frac{x\times x\times x\times x\times x\times x}{x\times x}=x\times x\times x\times x$$
$$=\mathbf{x^4}.$$

Example XIII. Divide $6x^3yz$ by $2x^2z$.

$$6x^3yz\div 2x^2z=\frac{6\times x\times x\times x\times y\times z}{2\times x\times x\times z}$$
$$=3\times x\times y=\mathbf{3xy}.$$

Example XIV. What is the square of x^4 ?

$$(x^4)^2=x^4\times x^4=\mathbf{x^8}.$$

Example XV. Find *a* value of $\sqrt{x^6}$.

$$x^3\times x^3=x^6;\quad \therefore\ \text{the square of } x^3 \text{ is } x^6.$$
$$\therefore\ a \text{ value of } \sqrt{x^6} \text{ is } \mathbf{x^3}.$$

Note. We shall see later that every number has two square roots, but at present we shall consider only one, as above.

EXERCISE II. *d*.

[Nos. 13-50 may be taken orally.]

Prove the following statements (cf. Examples X., XII. above).

1. $2^3 \times 2^4 = 2^7$.
2. $a^3 \times a^4 = a^7$.
3. $x^8 \div x^2 = x^6$.
4. $(b^3)^2 = b^6$.
5. $\sqrt{b^8} = b^4$.
6. $(x^2)^3 = x^6$.
7. $2x^3 \times x = 2x^4$.
8. $12x^4 \div 3x = 4x^3$.
9. $3xy \times 5x^2y = 15x^3y^2$.
10. $\sqrt{49^3} = 7^3$.
11. $\sqrt[3]{x^6} = x^2$.
12. $\sqrt[3]{8a^3b^3} = 2ab$.

Write down the values of the following:

13. $6^3 \times 6^2$.
14. $a^3 \times a^2$.
15. $y^3 \times y^4$.
16. $5^4 \div 5^3$.
17. $x^4 \div x^3$.
18. $11^3 \div 11$.
19. $x^3 \div x$.
20. $x^3 \times x$.
21. $(7^3)^2$.
22. $(x^3)^2$.
23. $(ab^4)^2$.
24. $(b^3)^3$.
25. $\sqrt{3^4}$.
26. $\sqrt{c^4}$.
27. $\sqrt{b^{16}}$.
28. $\sqrt[3]{x^{27}}$.
29. $5 . 7^2 \times 7$.
30. $5x^2 \times x$.
31. $3c^2 \times 2c^3$.
32. $4ab \times a$.
33. $\dfrac{6 \times 7^3}{2 \times 7^2}$.
34. $6x^3 \div 2x^2$.
35. $3x^3 \times 3x^3$.
36. $3xy \times 2xz$.
37. $6a^3 \div a$.
38. $6x^6 \div 2x^2$.
39. $2x^2 \times 3x^4$.
40. $3y \div y$.
41. $xy^2 \div xy$.
42. $4a^3b \div 2ab$.
43. $12abc \times a^2b$.
44. $3ab \times 2abc$.
45. $6abc^3 \div 6bc^3$.
46. $a^2 \times 2bc^2$.
47. $4p^2q^2r^2 \div 2pqr$.
48. $12x^6y^2 \div 4x^3$.
49. $(4x^2y)^2$.
50. $(2xy^2)^3$.
51. $(4ab^2c)^2$.
52. $(2a^2b^3c^5)^2$.
53. $\sqrt[3]{64x^{27}}$.
54. $\sqrt{16x^{16}}$.
55. $(x^3)^3 \div x^3$.
56. $(a^2)^3 \times (a^3)^2$.
57. $\dfrac{x \times x^2 \times x^4}{x^5}$.
58. $\dfrac{6xy \times 2xy^2}{3x}$.
59. $\dfrac{x^4 \times x^4}{x^8}$.
60. $\dfrac{(x^3)^2}{(x^2)^3}$.
61. $(2x)^3 + (3x)^3$.
62. $x^2(3x^2)^2 + x^8$.
63. $\dfrac{x^6}{x^2} + 3x^3$.

EXTRA PRACTICE EXERCISES. E.P. 2.

Like and Unlike Terms.

Give short-hand forms for the expressions in Nos. 1-36.

If there is no shorter form, say so.

1. $u+2u$.
2. $6x-2x$.
3. $3a+3a$.
4. $3y+2+y$.
5. $pq+qp$.
6. $10t^2-5t^2$.
7. $3b+0$.
8. $5l^2-5l^2$.
9. $3m+2m$.
10. $2z+z+3z$.
11. $x^2+4x^2-5x^2$.
12. $2a^2b+ba^2$.
13. $7R-5r$.
14. $8st-4st$.
15. $4a^2-4a$.
16. $6A+3A-9A$.
17. $10v^3-v^3$.
18. $c+cd$.
19. $2x+3y-x+y$.
20. $4+4a+2f-2$.
21. $A+B+B+A+1$.
22. $A^2-3AB-5AB+A^2$.
23. $x^2y+xy^2+yx^2$.
24. $2u-v+u+3v+3$.
25. $3ab+2ba+a+b$.
26. $ab-ac+bc-ba+ca-cb$.
27. $t^2-2t+4+2t^2-3t-1$.
28. $4yz-3xz+zy-xy$.
29. $x^3-5x+4-x^2+6x-3$.
30. $p^2+q^2-p^2+q^2$.
31. $x^2-1-x-1$.
32. $2+3c^2+c^4-c^2+1$.
33. $aba+bab$.
34. $aabb+abab+baab$.
35. $4-6t+2-3t$.
36. $10x+10x^3-10x^2-10$.

Products and Quotients.

37. $x^3\times x^3$.
38. $x^6\div x^2$.
39. $3x\times 3y$.
40. $p^2\times p^3$.
41. $2t\times t^2$.
42. $3ab\times 3ab$.
43. $6y^6\div 2y^2$.
44. $6a^2b\div 2a$.
45. $3ab\times 2a$.
46. $4t^4\times 2t^2$.
47. $c\times 2c\times 3c$.
48. $8r^2s^2\div 2rs^2$.
49. $2x^2\times 3xy^2$.
50. $10p^2\div p^2$.
51. $8p^4\div p$.
52. $3ab^2\times 3a^2b$.
53. $12x^6y^6\div 3x^2y^2$.
54. $2p\times 3q\times pq$.
55. $c^3\times c^3\div c^3$.
56. $z^2\times z^4\div z^6$.
57. $2x^2\times 3xy\times 4y^2$.
58. The square of $4ab^2$.
59. A square root of $4a^8$.

60. The cube of $3x^2$.
61. The square of $3x^3$.
62. A square root of $16x^2y^4$.
63. The cube of $2y^3z^2$.
64. The cube root of $8a^6$.
65. The square of $5pq^2r^3$.
66. $(2a)^2+(3a)^2$.
67. $(4b)^3\div(2b)^2$.
68. $(3c)^2\times 3c^2$.
69. $(p^4)^2\div p^2$.
70. $(2pq)^3\times 3q$.
71. $(xy^2)^2\times 2x^3$.
72. $(3t)^3-(2t)^3$.
73. $(2v)^2\times(3v)^3\div 6v^2$.
74. $(4z^2)^2\div 4z$.
75. $(3ab^3)^2\div 3ab$.
76. $x\times(2x)^2\times(3x)^3$.
77. $a^2\times(2ab)^2$.
78. $t(2t)^3\div t^2$.
79. $(2p^2)^3\div(2p^3)^2$.
80. $4x^3y^3z^3\div 4xyz$.

H.C.F. and L.C.M.

Of the two expressions, $4x^2y$ and $6xy^2$, there are several common factors, viz.: $\mathbf{2xy}$, 2, $2x$, $2y$, xy. But each of the last four is a factor of the first, $2xy$.

$\mathbf{2xy}$ is called the **Highest Common Factor (H.C.F.)** of $4x^2y$ and $6xy^2$.

Of the two expressions, $4x^2y$ and $6xy^2$, we can write down as many common multiples as we like, *e.g.*: $\mathbf{12x^2y^2}$, $120x^3y^4$, $24x^4y^2$, $36x^2y^5$, $48x^3y^3z^2$, etc. But each of the last four (and in fact every other common multiple) is a multiple of the first, $12x^2y^2$.

$12x^2y^2$ is called the **Lowest Common Multiple (L.C.M.)** of $4x^2y$ and $6xy^2$.

Example XVI. Find (i) the H.C.F., (ii) the L.C.M. of $6a^2b^2$ and $15ab^2c$.

(i) 3 is the H.C.F. of the coefficients 6, 15.

Of the powers of a, viz. a^2 and a, we take the lowest, a; of the powers of b, viz. b^2 and b^2, we take the lowest, b^2; we leave out c, because it is not a factor of $6a^2b^2$.

$$\therefore \text{ the H.C.F.} = 3\times a\times b^2 = \mathbf{3ab^2}.$$

As a check, $\dfrac{6a^2b^2}{3ab^2}=2a$; $\dfrac{15ab^2c}{3ab^2}=5c$. Therefore $3ab^2$ is a common factor, and there is no common factor of $2a$ and $5c$.

(ii) 30 is the L.C.M. of the coefficients 6, 15.

Of the powers of a, viz. a^2 and a, we take the highest a^2; of the powers of b, the highest is b^2, and of the powers of c, the highest is c.

$$\therefore \text{ the L.C.M.} = 30\times a^2\times b^2\times c = \mathbf{30a^2b^2c}.$$

As a check, $\dfrac{30a^2b^2c}{6a^2b^2}=5c$; $\dfrac{30a^2b^2c}{15ab^2c}=2a$. Therefore $30a^2b^2c$ is a common multiple, and there is no common factor of $5c$ and $2a$.

Exercise II. e. (Oral.)

1. Is 12 a multiple of (i) 4, (ii) 48 ?

2. Is 18 a multiple of (i) 180, (ii) 6 ?

3. Is 24 a factor of (i) 12, (ii) 48 ?

4. Is 24 a common multiple of (i) 2 and 3, (ii) 48 and 240 ?

5. Is 12 a common factor of (i) 3 and 4, (ii) 24 and 36 ?

6. Of which of the numbers 5, 6, 60, 10, 90, 120, 15 is 30 a factor ?

7. Of which of the numbers 5, 6, 60, 10, 90, 120, 15 is 30 a multiple ?

8. Of which of the expressions 2, $2x$, $4x^3$, $2xy$, $6xy^2$, $8x^4y$, $2x^2y^2$, x^4, $6x^3y$ is $2x^2$ a factor ?

9. Of which of the expressions $3x$, $2x^2$, $6xy^2$, $6y^2$, $12x^2y$, 2, $6x$, $48x^2y^2$ is $6x^2$ a multiple ?

10. Of which of the pairs of expressions (i) $12x^2$ and $6x^3$; (ii) $12xy$ and $4x^2y$; (iii) $10x^4y^4$ and $12x^5y^2$ is $2x^2$ a common factor ?

11. Of which of the pairs (i) $3x^2y$ and $4xy^2$; (ii) $12x^2$ and y^3; (iii) $10x$ and $4y$; (iv) $3x^2$ and y^4 is $12x^2y^3$ a common multiple ?

12. Is $20x^2y^3$ a multiple of $5y^2$? Simplify $\frac{20x^2y^3}{5y^2}$.

13. Is $3xyz$ a factor of $x^3y^3z^3$? Why ?

14. Is $4a^2b^3$ a common factor of $4a^2$ and b^3 ? Why ?

15. Is $3x^2y^2z^2$ a common multiple of $3xy$ and yz ? Why ?

16. Of the expressions, 2, $2x$, $4x^2$, $3x$, $12x^3y^3$, $6x^2y^2$, $2xy$, $6x^2y$, $18x^4$, x, xy, $6xy$, $3x^2y$, $2xz$, $6x^2z$, $6xyz^2$, write down those which are (i) factors of $2x^2$, (ii) factors of $3xy$, (iii) common factors of $2x^2$ and $3xy$, (iv) multiples of $2x^2$, (v) multiples of $3xy$, (vi) common multiples of $2x^2$ and $3xy$.

17. Of the expressions 6, a^2b, $a^2b^2c^2$, $3a^2$, b, $6a^4$, $6a^3bc$, abc^2, a, $3bc$, $2a^3$, $12a^3bc$, bc, write down those which are factors of (i) abc, (ii) $6a^3$.

Find the following:

18. H.C.F. of $2a$ and $6ab$.
19. L.C.M. of $2a$ and $6ab$.
20. H.C.F. of $4xy$ and $4xz$.
21. H.C.F. of b^2c^2 and $4bc$.
22. L.C.M. of $2ab$ and $3a^2b$.
23. L.C.M. of a, bca^2.
24. H.C.F. of ab and abc.
25. L.C.M. of $3a^2b^2$, $2a^3b^3$.
26. H.C.F. of $2a^2$, $6ab$, $4ac$.
27. L.C.M. of a^2, $6ab$, $3ab^3$.
28. H.C.F. of $10x^2$ and $15x^3$.
29. L.C.M. of $10x^2$ and $15x^3$.
30. H.C.F. and L.C.M. of 24, $3x$, $3xy$, $6xz$.

31. Find the H.C.F. of $12a^3b$, $9abc$, $15a^2b^3$, $24ab^2x$; check your answer as in Example XVI. above.

32. Find the L.C.M. of $4x^2y$, $6x^3z^3$, $3xyz^2$, $2z^4$; check your answer as in Example XVI. above.

33. Write down three expressions having $12a^2b^2$ as a common factor.

34. Write down three expressions of which $12a^2b^2$ is a common multiple.

Fractions.

Expressions involving fractions.

Example XVII. (i) There are 240 pupils in a school: of these, 163 are less than 11 years old, what fraction of the school is more than 11 years old?

(ii) There are n pupils in a school: of these, there are k pupils less than 11 years old, what fraction of the school is more than 11 years old?

(i) Out of 240 pupils, 163 pupils are under eleven.

$\therefore$ $240 - 163 = 77$ pupils are over eleven.

$\therefore$ the fraction of the school over eleven is $\frac{77}{240}$.

(ii) Out of n pupils, k pupils are under eleven.

$\therefore$ $n - k$ pupils are over eleven.

$\therefore$ the fraction of the school over eleven is $\dfrac{n-k}{n}$.

Exercise II. *f*.

1. What fraction is (i) 4 pence of 6 pence, (ii) x pence of y pence, (iii) p pence of s shillings?

2. Express (i) 7 inches in yards, (ii) l inches in yards, (iii) k yards in inches.

3. Express p shillings q pence (i) in pence, (ii) in shillings, (iii) in £.

4. The Sun rises at x a.m. and sets at y p.m.; how many hours long is the day? What fraction is this of 24 hours?

5. (i) A school contains 400 pupils; of these, 240 are boys; what fraction of the school are girls?

(ii) A school contains p pupils: of these, b are boys; what fraction of the school are girls?

6. From a stick l inches long, a portion p inches long is cut off. What fraction of the stick remains?

7. Find, with the data of Fig. 61, the values of

$$\text{(i) } \frac{AB}{BC}, \quad \text{(ii) } \frac{BC}{AC}, \quad \text{(iii) } \frac{AC}{AB}.$$

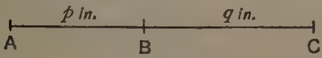


FIG. 61.

8. A shed l ft. long, b ft. wide (see Fig. 62), is built in the corner of a rectangular enclosure, x ft. long, y ft. wide. What

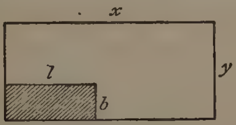


FIG. 62.

fraction of the total area does the shed occupy? What is the value of this fraction if (i) $x=2l$, $y=2b$, (ii) $x=2l$, $y=3b$?

9. How many $1\frac{1}{2}$d. stamps can be bought for (i) 1s., (ii) y shillings? What fraction of y shillings is $1\frac{1}{2}$d.?

10. Taking 8 Km. as equal to 5 miles, express (i) l Km. in miles, (ii) s miles in Km.

11. Some sweets are sold at 5d. for 2 oz. What is the cost of n oz.? What amount is obtained for p shillings?

12. What fraction is (i) 5 inches of 5 feet, (ii) l inches of l feet, (iii) $3l$ inches of l feet?

13. (i) If a man walks 4 miles an hour, how long does he take to walk p miles, q miles, $p+q$ miles ?

(ii) If a man cycles at 12 miles an hour, how long does he take to cycle 3 p miles ?

(iii) Simplify $\frac{3p}{12}$; $\frac{5p}{20}$.

14. Simplify (i) $\frac{8p}{12}$; (ii) $\frac{3p}{12}$; (iii) $\frac{2p}{3}+\frac{p}{4}$; (iv) $\frac{2p}{3}-\frac{p}{4}$.

15. If in Fig. 61, AB is x yards, BC is y yards, express the lengths of AB and BC also in feet and in inches; then express the fraction $\frac{AB}{BC}$ in three different ways.

16. How many parts, each $\frac{2}{3}$ inch long, can be cut off along a line (i) 4 inches long, (ii) b inches long ? Simplify $b \div \frac{4}{7}$.

17. The price of coal rises from xs. per ton to ys. per ton; how many less tons can now be bought for £P ?

18. A tap can fill a bath in t minutes. What fraction of the bath is filled in (i) 3 minutes, (ii) half a minute ?

19. Fine parallel lines are engraved on a bar at intervals of $\frac{1}{x}$ of an inch. There are n lines in all. What is the distance between the first line and the last ? [Invent a numerical example.]

20. (i) A sheet of paper is $\frac{1}{100}$ inch thick. How many sheets are there in a pile 2 inches thick ?

(ii) A sheet of paper is $\frac{1}{t}$ inches thick. How many sheets are there in a pile h inches high ?

Simplification of Fractions.

Example XVIII. Simplify $\frac{4x^3y}{6x^2yz}$.

The H.C.F. of the numerator and denominator is $2x^2y$.
Divide the numerator and denominator by $2x^2y$.

$$\therefore \frac{4x^3y}{6x^2yz}=\frac{2x^2y \times 2x}{2x^2y \times 3z}=\frac{2x}{3z}.$$

Exercise II. g.

Simplify the following fractions :

1. $\frac{2x}{2y}$. 2. $\frac{a}{a^2}$. 3. $\frac{2x}{2}$. 4. $\frac{6x}{9x}$. 5. $\frac{ab}{ac}$.

6. $\frac{3ab}{a}$. 7. $\frac{x^2}{xy}$. 8. $\frac{xyz}{xyz}$. 9. $\frac{5a^2}{15ab^2}$. 10. $\frac{2a}{2ab}$.

11. $\frac{3xyz}{30z^2}$. 12. $\frac{a}{abc}$. 13. $\frac{cd}{cbd}$. 14. $\frac{6a^3}{6a^2b}$. 15. $\frac{4x^3yz}{4yz}$.

16. $\frac{4a^6b}{2a^3c}$. 17. $\frac{a^4}{a^{16}}$. 18. $\frac{6a^2b^2}{8a^2bc}$. 19. $\frac{a^2x}{xa^2}$. 20. $\frac{6a^2cd^2}{9abc^2}$.

21. $\frac{a^2 . (ab)^3}{b^2 . (ab)^2}$. 22. $\frac{x^2yxz^3}{xy^2z}$. 23. $\frac{a^2}{a^3b}$. 24. $\frac{a^2b^3c^2}{a^2b^3c^2}$.

25. $\frac{2ac}{a^2c^2}$. 26. $\frac{18b^3c^3}{12bc^2}$. 27. $\frac{x^{10}y^8z^4}{x^5y^4z^2}$.

28. $\frac{3ab^2}{6b^2a^2}$. 29. $\frac{a^{100}}{a^{10}}$. 30. $\frac{x^{20} . x^{30}}{x^{60}}$.

Example XIX. Express $\frac{2a}{3b}$ in the form $\frac{?}{6b}$.

$$\frac{2a}{3b}=\frac{2a \times 2}{3b \times 2}=\frac{4a}{6b}.$$

Example XX. Express $\frac{x^2}{5yz}$ in the form $\frac{?}{10x^2y^2z^2}$.

$$10x^2y^2z^2 = 5yz \times 2x^2yz.$$

$$\therefore \frac{x^2}{5yz}=\frac{x^2 \times 2x^2yz}{5yz \times 2x^2yz}=\frac{2x^4yz}{10x^2y^2z^2}.$$

Example XXI. Add together $\frac{2}{x}$ and $\frac{1}{2x}$.

The L.C.M. of the denominators x and $2x$ is $2x$.
Express each fraction so that its denominator is $2x$:

$$\frac{2}{x}+\frac{1}{2x}=\frac{4}{2x}+\frac{1}{2x}=\frac{4+1}{2x}=\frac{5}{2x}.$$

Compare this with $\frac{2}{7}+\frac{1}{14}=\frac{4}{14}+\frac{1}{14}=\frac{5}{14}$.

Example XXII. Subtract $\frac{b}{4a}$ from $\frac{a}{6b}$.

The L.C.M. of the denominators $4a$, $6b$ is $12ab$.

$$\frac{a}{6b}-\frac{b}{4a}=\frac{a\times 2a}{6b\times 2a}-\frac{b\times 3b}{4a\times 3b}=\frac{2a^2}{12ab}-\frac{3b^2}{12ab}=\frac{2a^2-3b^2}{12ab}.$$

Note. As soon as the method is understood, the working would merely be $\frac{a}{6b}-\frac{b}{4a}=\frac{2a^2-3b^2}{12ab}$.

Example XXIII. Simplify $a-\frac{b^2}{a}$.

Since $a=\frac{a}{1}$, the L.C.M. of the denominators 1, a is a.

$$\therefore\ a-\frac{b^2}{a}=\frac{a^2}{a}-\frac{b^2}{a}=\frac{a^2-b^2}{a}.$$

Exercise II. *h*.

Fill in the gaps in Nos. 1-7.

1. $\frac{2}{3}=\frac{\quad}{6}=\frac{\quad}{30}=\frac{\quad}{42}$.

2. $\frac{x}{y}=\frac{\quad}{2y}=\frac{\quad}{ay}=\frac{\quad}{y^2}$.

3. $\frac{2x}{5y}=\frac{\quad}{10y}=\frac{\quad}{5yz}=\frac{\quad}{y^2}$.

4. $b=\frac{\quad}{3}=\frac{\quad}{a}=\frac{\quad}{b}$.

5. $\frac{3a^2b}{6ab^2}=\frac{\quad}{12a^2b^2}=\frac{\quad}{2b}$.

6. $\frac{b}{c}=\frac{\quad}{c^2}=\frac{\quad}{3bc}=\frac{b^3}{\quad}$.

7. (i) $\frac{2x}{y}=\frac{\quad}{3y^2}$; (ii) $a^3=\frac{\quad}{b^2}$; (iii) $\frac{1}{x}=\frac{y^2}{\quad}$; (iv) $\frac{a^2}{x}=\frac{a^3b^3}{\quad}$.

8. Express the fractions $\frac{5}{2x}$, $\frac{x}{6y^2}$, $\frac{y}{3x^2}$, so that all their denominators are equal. What is the simplest way?

Simplify the following fractions:

9. $\frac{2}{7}+\frac{3}{7}$; $\frac{2}{a}+\frac{3}{a}$; $\frac{3}{4a}+\frac{1}{4a}$; $\frac{5}{3a}-\frac{1}{3a}$.

10. $\frac{1}{7}+\frac{1}{2\times 7}$; $\frac{1}{x}+\frac{1}{2x}$; $\frac{y}{x}+\frac{3y}{5x}$; $\frac{2y}{x}-\frac{3y}{4x}$.

11. $\frac{2}{5}+\frac{1}{10}$; $\frac{2}{a}+\frac{1}{2a}$; $\frac{3b}{2a}+\frac{b}{a}$; $\frac{b}{a}-\frac{b}{2a}$.

12. $\frac{1}{2x}+\frac{1}{3x}$.

13. $\frac{y}{x}+\frac{x}{y}$.

14. $\frac{5}{a}-\frac{2}{a}$.

15. $\frac{1}{x}-\frac{1}{3x}$.
16. $1+\frac{2}{3}$; $1+\frac{a}{b}$.
17. $1-\frac{2}{7}$; $1-\frac{a}{b}$.

18. $\frac{1}{4}+3$; $\frac{1}{x}+3$.
19. $\frac{1}{ab}+\frac{2}{a}$.
20. $1-\frac{1}{xy}$.

21. $\frac{1}{x}-\frac{x}{11}$.
22. $\frac{1}{xy}-\frac{xy}{z}$.
23. $7-\frac{2}{x}$.

24. $ab-\frac{c}{d}$.
25. $a+\frac{1}{b}$.
26. $\frac{a}{b}+\frac{2a}{3b}$.

27. $\frac{x}{2y}+\frac{x}{6y}$.
28. $\frac{a}{2b}-\frac{a}{6b}$.
29. $\frac{a^2}{b^2}-\frac{a}{b}$.
30. $a-\frac{2}{3}a$.

31. Subtract $\frac{x^2}{3}$ from $\frac{x^2}{2}$.
32. Add $\frac{x}{y}$ to 1.

33. Add $\frac{a}{b}$ to $\frac{c}{b}$.
34. Subtract $\frac{1}{12x}$ from $\frac{1}{4x}$.

35. Add $\frac{x}{2}$ to $\frac{1}{x}$.
36. Subtract $\frac{a}{b}$ from $\frac{a}{c}$.

37. $\frac{2}{7}\times 3$; $\frac{a}{b}\times 3$; $\frac{x}{y}\times z$; $\frac{2x}{y}\times\frac{z}{4}$.

38. $\frac{2}{7}\div 3$; $\frac{a}{b}\div 3$; $\frac{x}{y}\div z$; $\frac{x}{3y}\div\frac{z}{6}$.

39. $3\times\frac{a}{b}$.
40. $\frac{2}{3}\times\frac{x}{y}$.
41. $\frac{2}{3}\div\frac{x}{y}$.
42. $3\div\frac{a}{b}$.

43. $\frac{a}{b}\times\frac{2}{3}$.
44. $\frac{a}{b}\div\frac{2}{3}$.
45. $z\div\frac{x}{y}$.
46. $\frac{x^2}{y^2}\times\frac{y}{x}$.

47. $\dfrac{a}{\frac{b}{c}}$.
48. $\dfrac{\frac{a}{b}}{c}$.
49. $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.
50. $\dfrac{\frac{1}{x}}{\frac{1}{y}}$.

51. $\frac{a}{b}\times\frac{c}{a}$.
52. $\frac{x}{y}\div\frac{x}{z}$.
53. $\frac{a^2}{bc}\times\frac{b^2}{ac}$.
54. $b\div\frac{b}{c}$.

55. $xz\div\frac{x}{z}$.
56. $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{a}$.
57. $x^3\div\frac{x^2}{y^2}$.
58. $\frac{ab}{c^2}\times\frac{ac}{b^2}$.

59. $\frac{a^2b^2}{c}\times\frac{c^2}{ab}$.
60. $\frac{10a}{b^2}\div\frac{2a}{3}$.
61. $\frac{12bc}{35a^2}\times\frac{14ab^2}{24b^3}$.
62. $x\div\frac{1}{y}$.

63. $a-\frac{a}{4}$; $x-\frac{7}{100}x$; $y+\frac{2}{9}y$; $b+\frac{5}{100}b$.

64. $\frac{2\frac{1}{2}}{100}x$.

65. $x+\frac{x}{y}$.

66. $\frac{a}{b}+\frac{a}{c}$.

67. $x+\frac{y^2}{x}$.

68. $\frac{x}{y}+1+\frac{y}{x}$.

69. $\frac{1}{xy}+\frac{1}{x}$.

70. $\frac{2}{x^2}+\frac{2}{x}$.

EXTRA PRACTICE EXERCISES. E.P. 3.

H.C.F. and L.C.M.

Find the following :

1. H.C.F. of $6x$, $3xy$.
2. H.C.F. of a^2b, ab^2.
3. L.C.M. of $2pq$, $6p^2q$.
4. L.C.M. of $3y^2$, $2yz$.
5. H.C.F. of $5a^2b$, $10ab^3$.
6. H.C.F. of $6c^3x^4$, $9c^2x^3$.
7. L.C.M. of $2xy$, $2xyz$.
8. L.C.M. of $4abxy$, $14byz$.

Find the H.C.F. and L.C.M. of the following :

9. $3x^2$, $2x^3$.
10. $5a^2b$, $10b^2$.
11. $6y^3$, $8y^2z$.
12. $4x^3$, $6y^3$.
13. $4pq$, $5r^2$.
14. $8yx$, $10xy$.
15. 12, $10t$.
16. y^6z^2, y^3z.
17. $9ef^2$, $6fg^2$.
18. x^2y, xy^2.
19. $4b^3$, $9c^3$.
20. $12b^2x$, $8aby$.
21. $10x^2$, $12xy$, $2x^3$.
22. $6a^3b^3$, $9a^2bc$, $12a^4b^2c^2$.
23. $2z^2$, $3z^3$, $4z^4$.
24. $4r^2$, $5rs$, $6s^2$.
25. $6x$, $2xy$, $2y^3$, $4x^2$.
26. $2x^2yz$, $3xy^2z$, $4xyz^3$.
27. 8, $10p$, $4p^3$, $3q^2$.
28. $9x^3y$, $6y^3z$, $12x^2y^2z^2$.
29. $10x^3$, $15abx^2$, $20cxy$.
30. $(2ax)^2$, $(6bx)^2$, $(2cx)^2$.

Fractions.

Simplify the following fractions ; if there is no simpler form, say so.

31. $\frac{a^2}{a}$.
32. $\frac{b^3}{b^2}$.
33. $\frac{3c}{3}$.
34. $\frac{4d}{4}$.
35. $\frac{5e}{5e}$.
36. $\frac{p^3}{3p}$.
37. $\frac{r^6}{r^2}$.
38. $\frac{s}{st}$.
39. $\frac{u}{u^2}$.
40. $\frac{v^3}{v^6}$.

41. $\frac{2xy}{2xz}$. 42. $\frac{2xy}{x^2}$. 43. $\frac{z}{z}$. 44. $\frac{a}{ab}$. 45. $\frac{c^2}{2cd}$.

46. $\frac{3ef}{3fe}$. 47. $\frac{4g^4}{h^4}$. 48. $\frac{kl}{k^3l^3}$. 49. $\frac{4m^4n^4}{mn}$. 50. $\frac{p^3}{3qr}$.

51. $\frac{6t^2u^2}{9u^2v^2}$. 52. $\frac{w^2}{w^2x^2}$. 53. $\frac{yz}{z}$. 54. $\frac{a^6b^6}{a^2b^2}$. 55. $\frac{c^6}{3c^2}$.

56. $\frac{(3d)^2}{(2d)^2}$. 57. $\frac{(2a^3b)^3}{a^3b^3}$. 58. $\frac{x^3}{(2x)^2}$. 59. $\frac{p^2}{(pqr)^2}$. 60. $\frac{x^2}{(yz)^2}$.

61. $\frac{1}{a}+\frac{1}{2a}$. 62. $\frac{1}{b}-\frac{1}{3b}$. 63. $\frac{c}{cd}-\frac{1}{d}$.

64. $e-\frac{1}{e}$. 65. $1+\frac{1}{f}$. 66. $gh+\frac{g}{h}$.

67. $\frac{2}{l}-\frac{l}{2}$. 68. $\frac{1}{2mn}+\frac{3}{n^2}$. 69. $\frac{p}{q}-\frac{p^2}{q^2}$.

70. $r-\frac{s}{t}$. 71. $\frac{1}{u}-\frac{v}{uv}$. 72. $\frac{x^3}{6x^6}-\frac{1}{6x^2}$.

73. $\frac{1}{6yz}+\frac{1}{4z^2}$. 74. $\frac{a}{10bc}-\frac{b}{15ac}$. 75. $\frac{3a}{2x^2}+\frac{2b}{3y^2}$.

76. $a\div\frac{a}{c}$. 77. $\frac{1}{b}\div\frac{1}{d}$. 78. $ef\div\frac{1}{e}$.

79. $1\times\frac{p}{q}$. 80. $\frac{r}{s}\div 1$. 81. $\frac{x^2}{y^2}\div\frac{y}{x}$.

82. $\frac{3}{4}xy\div\frac{6}{xz}$. 83. $\frac{ab}{ab}\div ac$. 84. $b^6\div\frac{1}{b^2}$.

85. $\left(\frac{2c^2}{d}\right)\div cd$. 86. $\frac{1}{e}\times\left(\frac{3e}{f}\right)^3$. 87. $3p^3\div\left(\frac{1}{2p}\right)^3$.

88. $\frac{z}{3yz}+\frac{x}{6xy}$. 89. $\left(p\div\frac{1}{p}\right)^3$. 90. $\frac{uv}{vu}+\frac{u^2v^2}{v^2u^2}$.

SUPPLEMENTARY EXERCISE. S. 3

1. If x per cent. of a school are boys, what fraction of the school are girls?

2. If $\frac{1}{a}$ of a stick is cut off, what fraction remains?

Simplify the following fractions :

3. $\dfrac{xy}{y^2}+\dfrac{z^2}{yz}$. 4. $\dfrac{n}{\frac{1}{2}}+\dfrac{n}{\frac{1}{3}}$. 5. $\dfrac{a}{\frac{1}{3}}-\dfrac{b}{\frac{1}{4}}$.

6. $\dfrac{p}{2\frac{1}{2}}+\dfrac{q}{1\frac{2}{3}}$. 7. $\dfrac{t}{0{\cdot}1}-\dfrac{t}{0{\cdot}2}$. 8. $\dfrac{u}{0{\cdot}75}+\dfrac{v}{0{\cdot}6}$.

9. Does the value of $\dfrac{3x}{4x}$ alter, if different values are given to x ? What about $\dfrac{3+x}{4+x}$?

10. Does the value of $\dfrac{2x+3}{10}-\dfrac{x}{5}$ depend on the value of x ?

11. If 8 Km. = 5 mi., what fraction is x Km. of (i) 1 mile, (ii) x miles ?

12. In Fig. 63, what fraction is (i) the shaded area of the total area, (ii) the shaded area of the unshaded area ?

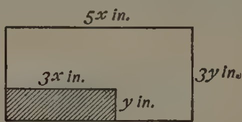


FIG. 63.

13. If, in No. 12, $x=\dfrac{3y}{4}$ express the perimeter of the shaded area as a fraction of the perimeter of the total area ?

14. By what must $\dfrac{a}{b}$ be multiplied to give $\dfrac{b}{a}$?

15. A train runs up an incline of 1 in n at x miles per hour (see Fig. 64); what vertical height in feet does it rise per second ?

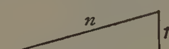


FIG. 64.

16. One tap fills a bath in x minutes and a second tap fills it in y minutes. What fraction of the bath is filled in 1 minute when both taps are running ?

17. A tank is filled by two pipes ; one delivers x gallons in a minutes, the other delivers y gallons in b minutes. How many gallons enter the tank per minute ?

18. After each stroke of an air exhaust-pump, $\frac{x}{y}$ of the air in the vessel before the stroke remains. What fraction remains after 3 strokes ?

19. The area of a rectangular field is $6x^2$ sq. yd. and its length is 10 xy yd. What is its breadth ?

20. A sheet of paper is $\frac{1}{x}$ inches thick. How many sheets are there in a pile x inches high ?

21. A tank x ft. long, y ft. wide, contains 1 cu. ft. of water. What is the depth of the water in inches ?

Simplification of Brackets.

In practice, expressions containing brackets are simplified by applying formal rules. But before these rules are committed to memory, they should be justified by *oral illustrations.*

Example XXIV. A possesses 12 shillings and owes B 5 shillings and owes C 3 shillings : how much has A left after the debts are paid ?

After A has paid B he has $(12-5)$ shillings left. He then pays C ; after that, he has $(12-5-3)$ shillings left.

If A pays B and C at the same time, he hands over $[5+3]$ shillings ; after that, he has $(12-[5+3])$ shillings left.

$$\therefore\ 12-[5+3]=12-5-3$$

or in general, $\mathbf{x-[y+z]=x-y-z}$.

Example XXV. A has 12 shillings in his left pocket and 3 shillings in his right pocket, and owes B 5 shillings : how much has A left after his debt is paid.

If A takes out of his left pocket the 12 shillings and pays B, $(12-5)$ shillings remains in his hand : there are also 3 shillings in his right pocket.

$\therefore$ A has $[(12-5)+3]=[12-5+3]$ shillings left.

If A takes out of his right pocket the 3 shillings and hands them to B, he still owes B $(5-3)$ shillings, and still has 12 shillings.

$\therefore$ after he has paid B the rest, he has $[12-(5-3)]$ shillings.

$$\therefore\ 12-(5-3)=12-5+3.$$

Or in general, $\mathbf{x-(y-z)=x-y+z}$.

Note. By $(x-y+z)$ we mean " from x subtract y, then add z to the result." Consequently $x-y+z$ is short for $(x-y)+z$.

These two examples illustrate the following results

$$\mathbf{x}-(\mathbf{y}+\mathbf{z})=\mathbf{x}-\mathbf{y}-\mathbf{z} \quad \text{and} \quad \mathbf{x}-(\mathbf{y}-\mathbf{z})=\mathbf{x}-\mathbf{y}+\mathbf{z},$$

where $x-y-z$ means $(x-y)-z$ and $x-y+z$ means $(x-y)+z$.

These results show how brackets with a minus sign in front of them may be removed.

It is easy to illustrate the other two cases.

Example XXVI. A boy has 2 shillings. He then receives from home two postal orders, one of 10s. and the other of 5s., so that he has altogether $(2+10+5)$ shillings.

But the result would be just the same if he had received one postal order for $[10+5]$ shillings, in which case he has $(2+[10+5])$ shillings.

$$\therefore\ 2+[10+5]=2+10+5.$$

Or in general, $\mathbf{x}+(\mathbf{y}+\mathbf{z})=\mathbf{x}+\mathbf{y}+\mathbf{z}$,

where $x+y+z$ means $(x+y)+z$, *i.e.* first add y to x, then add z to the result.

Example XXVII. A has 2 shillings; he then receives 10s. from B, and then pays a debt of 4s. to C.

As a result, he has $(2+10-4)$ shillings.

The result would be the same if B paid C the 4s. due to C, and handed the balance $[10-4]$ shillings to A.

$$\therefore\ 2+[10-4]=2+10-4.$$

Or in general, $\mathbf{x}+(\mathbf{y}-\mathbf{z})=\mathbf{x}+\mathbf{y}-\mathbf{z}$,

where $x+y-z$ means $(x+y)-z$, *i.e.* first add y to x, then subtract z from the result.

Exercise II. *j*. (Oral.)

1. What is the value of (i) $(20+13)-6$; (ii) $20+(13-6)$?

2. What is the value of (i) $(100+20)+9$; (ii) $100+(20+9)$?

3. What is the value of (i) $15-(3+2)$; (ii) $(15-3)-2$?

4. What is the value of (i) $13-(12-1)$; (ii) $(13-12)+1$?

5. A stick is x inches long. A man first cuts off a length of y inches and after that a length of z inches. What is the length of what is left? What would be the answer if he had cut off a length of $(y+z)$ inches by a single cut?

6. A man has £x in the Bank, he draws out £y and then hands back £z to the Bank. How much has he then in the Bank? What is the result if he merely draws out £$(y-z)$?

7. A man motoring north from London travels x miles in the morning, y miles in the afternoon, z miles in the evening. How far is he from London at the end of the day? How far would he be if he had travelled x miles in the morning, $(y+z)$ miles in the afternoon and nothing in the evening?

8. There are x shillings in a cash box; a boy then puts in y shillings and the next day takes out z shillings. How much is left in the cash box? Is the result the same as putting in $(y-z)$ shillings and taking nothing out?

Example XXVIII. A room is h ft. high, l ft. long, b ft. broad. Find the total area of the four walls.

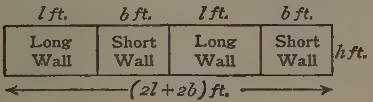


FIG. 65.

Figure 65 represents the four walls, folded out flat.

The perimeter of the floor of the room

$$=(l+b+l+b)=(2l+2b) \text{ ft.}$$

$\therefore$ the total area of the four walls $=(2l+2b)\times h$ sq. ft.

$$=h(2l+2b) \text{ sq. ft.}$$

But, by taking each wall in turn, we see that the area is

$$(hl+hb+hl+hb)=(2hl+2hb) \text{ sq. ft.}$$

$\therefore$ the area is represented by either of the expressions

$$h(2l+2b) \text{ sq. ft.} \quad \text{or} \quad (2hl+2hb) \text{ sq. ft.}$$

We may also argue as follows:

One long wall and one short wall form a rectangle h ft. high and $(l+b)$ ft. long.

$\therefore$ half the total area $=h(l+b)$ sq. ft.

$\therefore$ the total area $=2h(l+b)$ sq. ft.

We therefore see that

$$2h(l+b)=h(2l+2b)=2hl+2hb,$$

so that $2h(l+b)$ is simply a short-hand form for $2hl+2hb$.

The examples in Exercise II. *j.* illustrated four rules which should in future be assumed.

(i) $\mathbf{x+(y+z)=x+y+z}$; (ii) $\mathbf{x+(y-z)=x+y-z}$;

(iii) $\mathbf{x-(y+z)=x-y-z}$; (iv) $\mathbf{x-(y-z)=x-y+z}$.

The argument in Example XXVIII. showed that when an expression in a bracket is multiplied by a number, each term in

the bracket must be multiplied by that number when the bracket is removed. Hence

(v) $(a+b)c = c(a+b) = ac + bc$; (vi) $(a-b)c = c(a-b) = ac - bc$.

And from (v) and (vi), we have

(vii) $(ac+bc) \div c = a+b$; (viii) $(ac-bc) \div c = a-b$.

These equalities (i)-(viii) are summed up in the following rules:

(I). If an expression in a bracket is multiplied (or divided) by a number, placed outside the bracket, each term in the bracket must be multiplied (or divided) by that number when the bracket is removed.

(II). If a bracket has a + sign in front of it, the sign of each term in the bracket remains the same when the bracket is removed.

(III). If a bracket has a − sign in front of it, the sign of each term in the bracket is changed when the bracket is removed.

Example XXIX. Simplify $(2a+5b)-(a-2b)$.

$$(2a+5b)-(a-2b) = 2a+5b-a+2b = \mathbf{a+7b}.$$

Example XXX. Multiply $2x-5y$ by $3x$.

$$(2x-5y)3x = 2x \times 3x - 5y \times 3x = \mathbf{6x^2 - 15xy}.$$

Example XXXI. Simplify $3(2p-5q)-2(p+3q)$.

$$3(2p-5q)-2(p+3q) = (6p-15q)-(2p+6q)$$
$$= 6p-15q-2p-6q = \mathbf{4p-21q}.$$

Note. To avoid mistakes, it is advisable to multiply in one line and remove the brackets in the next line. *Do not try to do both operations in a single line.*

Exercise II. *k*.

Simplify the following:

1. $x+(y+2z)$.
2. $x+(2y-z)$.
3. $x+(x+y)$.
4. $x+(y-x)$.
5. $x-(y-x)$.
6. $(x+1)+(x-1)$.
7. $(x+2)-(x+1)$.
8. $x^2-(2x-1)$.
9. $1-(x-1)$.
10. $a-(b+2c)$.
11. $a-(2b-c)$.
12. $2a-(a+b)$.
13. $(a+b+c)+(a-b-c)$
14. $(a+b+c)-(a-b-c)$.
15. $x^3+x^2-(x+1)$.
16. $(x^3+x)-(x^2+1)$.
17. $(x^2+2xy+y^2)-(x^2-2xy+y^2)$.
18. $(3x^2-4x+2)-(2x^2-x-3)$.
19. $(x-z)-(y-z)$.
20. $(b-c)+(c-a)+(a-b)$.
21. $x^2+3y^2+z^2-(3y^2-z^2)$.

22. Evaluate 871×99 by writing it as $871(100-1)$.

23. Evaluate $63 \times 1\frac{6}{7}$ by writing it as $63(2-\frac{1}{7})$.

24. $(x+y)z$. 25. $z(x+y)$. 26. $(x-y)z$.

27. $z(x-y)$. 28. $2a(b-c)$. 29. $(2x-y)2x$.

30. $x(x^2-2x+1)$. 31. $(xz+yz)\div z$. 32. $(a^2-ab)\div a$.

33. $(x^2-2x)\div x$. 34. $(2x^3-3x)\div x$. 35. $(a^2-ab-ac)\div a$.

36. $2xy(x-y)$. 37. $(a^2-a)\div a$. 38. $a^2-a(a-b)$.

39. $a(a-b)+ab$. 40. $x(x-1)+2x$.

41. $x(2x-1)+2(x-1)$. 42. $(x-y).x+(x+y).y$.

43. $3(2x-5y)-2(x+4y)$. 44. $(10x^3-15x^2y)\div 5x$.

45. $(8x^2y^2+6xy)\div 2xy$. 46. $4x(x+2y)-3x(x-y)$.

47. $x\left(1-\frac{1}{x}\right)$. 48. $\left(x-\frac{1}{y}\right).y$.

49. $ab\left(\frac{1}{a}+\frac{1}{b}\right)$. 50. $x-\frac{1}{x}(x^2-1)$.

51. $5(2a-3b+4c)-3(a-2b-6c)$.

52. $a(b-c)+b(c-a)+c(a-b)$.

53. If $y=7x+19$, find the increase in the value of y due to an increase of 3 in the value of x.

54. If v and d are connected by the formula

$$v=0{\cdot}209d\div(1+0{\cdot}25\sqrt{d}),$$

what is the error per cent., to one significant figure, in taking v as $1{\cdot}6$ when $d=16$?

It is sometimes useful to have more than one set of brackets in an expression.

Example XXXII. A book is marked for sale at 7 shillings, but there is a discount of 1s. 9d. for cash; a man orders 6 copies to be sent him by post (postage 10d.); express by brackets the change he will receive out of £2.

The net price of each book is 7s. – 1s. 9d.

∴ the parcel of 6 books with the postage costs

6(7s. – 1s. 9d.) + 10d.

∴ the change he receives is

£2 – (6(7s. – 1s. 9d.) + 10d.).

To prevent confusion, it is better, when one set of brackets contains another between them, to choose brackets of different shapes.

The above answer should be written in the form

$$£2 - [6(7s. - 1s.\ 9d.) + 10d.].$$

The following examples show the usual forms of brackets employed: $a-(b+c)$; $a-[b+c]$; $a-\{b+c\}$; $a-\overline{b+c}$. Compare the use of the line in the last expression with the line in $\frac{b+c}{3}$, which means $\frac{1}{3}(b+c)$.

Example XXXIII. State in words the process of simplifying

$$a-\{b(c+d)+e\}.$$

Multiply the sum of c and d by b.

Add e to this result and then subtract the whole from a.

Example XXXIV. Subtract $x-(y-2z)$ from $1+2(z-x)$.

The result of the subtraction equals

$$[1+2(z-x)]-[x-(y-2z)]$$
$$=[1+(2z-2x)]-[x-y+2z]=1+2z-2x-x+y-2z$$
$$=1-3x+y.$$

Exercise II. *l*.

1. (i) Show that £3 15s. 7d. $=£[\{(3\times 20+15)12+7\}\div 240]$.
   (ii) In the same way, express £x ys. zd. in £.

2. (i) Show that
   13 ton 7 cwt. 1 qr. $=[\{(13\times 20+7)4+1\}\div 80]$ tons.
   (ii) In the same way, express a ton b cwt. c qr. in tons.

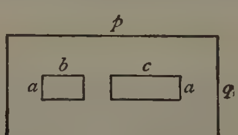


Fig. 66.

3. There are two holes in a plate, as shown in Fig. 66; all the corners are right-angled and the units are inches. Express in bracket-form as simply as possible the area of the plate.
   How could you write the result if $q=3a$?

4. Find in bracket-form as simply as possible the area of

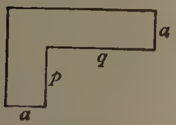


FIG. 67.

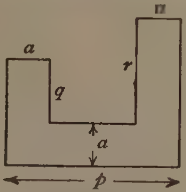


FIG. 68.

(i) Fig. 67, where all the corners are right-angled ;

(ii) Fig. 68, where all the corners are right-angled ;

(iii) Fig. 69, which is formed of a rectangle with triangles on one pair of opposite sides as bases.

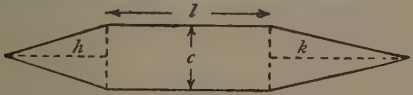


FIG. 69.

The units in the figures are inches.

5. Find the area of the triangle in Fig. 70, the figure enclosing it being a rectangle and the units being inches.

6. A path d ft. wide surrounds a lawn which is l ft. long, b ft. wide. Express in two different ways the area of the path.

7. The total surface area of a closed circular cylinder, of radius r in., height h in., is approximately $\frac{44}{7}r(h+r)$ sq. in. What is the total surface area of a cylinder 1 inch high and 1 foot in diameter ?

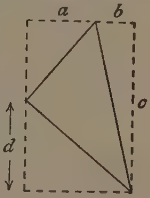


FIG. 70.

8. If a wave h ft. high travels across the sea where the depth is d ft., its velocity is $\sqrt{[32(d+3h)]}$ ft. per sec. What is the velocity of a wave 2 ft. high passing across water 12 ft. deep ?

9. The total surface area of a closed circular cone, of base-radius r cm., height h cm., is approximately

$$\frac{22r}{7}[r+\sqrt{(h^2+r^2)}].$$

What is the total surface area of a cone, (i) of height 4 in. and base-diameter 6 in., (ii) of height p in. and base-diameter $\frac{3p}{2}$ in.?

10. If two totally inelastic bodies of masses W, w lb. collide when moving in the same direction with velocities V, v ft. per sec. respectively, the loss of energy is $\frac{W \cdot w}{64}(V-v)^2 \div (W+w)$ ft.-lb. Find the loss of energy if an inelastic body of mass 12 lb. travelling at 30 ft. per sec. overtakes an inelastic body of mass 8 lb. travelling at 20 ft. per sec.

11. The peaks A, B, C of three mountains are in line and at heights a, b, c feet above sea-level respectively; if C is just visible from A, the distance of C from A is approximately $\{\sqrt{[\frac{3}{2}(a-b)]}+\sqrt{[\frac{3}{2}(c-b)]}\}$ miles. What is the distance of C from A if the heights of A, B, C are 2200, 1600, 2950 feet respectively?

12. State in words the process of simplifying:

(i) $a+[b-(c+d)]$; (ii) $(x-y)-(p-q)$;

(iii) $[u(v+w)-s]\div t$; (iv) $e\div\{f-(l-m)\}$.

13. In a brick wall l ft. long, h ft. high, there are two windows of height a ft. and of lengths u ft., v ft. respectively; there are also two windows of height b ft. and of lengths p ft., q ft. respectively. Express by brackets the area of the brickwork and then remove all brackets.

Express by brackets the statements in Nos. 14-18 and then simplify the results.

14. Subtract $a-(b-c)$ from $b-(a-c)$.

15. Add $3p+2(q-r)$ to $4q-3(p-r)$.

16. The excess of $x^2+y(2x-y)$ over $x(x+y)-2y^2$.

17. The amount that must be added to $2f-3(u-v)$ to give $3u-4(v-f)$.

18. The amount that must be subtracted from $5a(a-b)-b^2$ to leave $b^2-3a(b-a)$.

Simplify the expressions in Nos. 19-28.

19. $a+b-\{a-(b+c)\}$.
20. $2[\overline{x+y}-\overline{x-y}]$.
21. $3[4x-2(x-y)]$.
22. $3v-\{u+2(v-u)\}$.
23. $2p[p-\overline{p-q}-q]$.
24. $3(f-2g)-2(f-g)$.
25. $4(b^2-5bc+c^2)-3c(2b+c)$.
26. $W-W[2-\overline{1+e}]$.
27. $[a(a-4b)+7ac]\div a$.
28. $\{a^2(1-a)+b(a+a^2)\}\div a$.

Fill in the gaps in Nos. 29-34.

29. $a-b-c=a-(\qquad)$.
30. $u-v+w=u-(\qquad)$.
31. $x+2y-2z=x+2(\qquad)$.
32. $p-2q-2r=p-2(\qquad)$.
33. $c+3d-4e=c+3(\qquad)$.
34. $g-3h-1=g-(\qquad)$.

EXTRA PRACTICE EXERCISES. E.P. 4.

Brackets.

Simplify the following :

1. $a+(b-3c)$.
2. $d+(b-d)$.
3. $2f-(f+g)$.
4. $(k+l)-(k-l)$.
5. $p+2q-(3q-p)$.
6. $(r-2s)-s$.
7. $3t+(t-2s)$.
8. $4y-(x-3y)$.
9. $(x-y)+(y-x)$.
10. $x^2-(x^2-y^2)$.
11. $(a-2z)-(z-2a)$.
12. $1+b-(1-b)$.
13. $2(c-d)-(c+d)$.
14. $3(f+h)-2(h-f)$.
15. $5-2(1-k)$.
16. $p-3(r-p)$.
17. $2q+5(a+2q)$.
18. $4(u-v)-3(u+v)$.
19. $x(x+y)-x(x-y)$.
20. $x(y+x)-y(y+x)$.
21. $yz-z(y-z)$.
22. $(x^2-5x)\div x$.
23. $(4a^2-6ab)\div 2a$.
24. $c-c\left(1-\frac{1}{c}\right)$.
25. $4e(e-2f)-3f(e-2f)$.
26. $a^3(a^3+2b^3)-b^3(a^3-2b^3)$.
27. $3(x+y-2z)-3(x-y+2z)$.
28. $x(y-z)+y(z-x)+z(x-y)$.
29. $(6x^2y-8xy^2)\div 2xy$.
30. $x\left(1+\frac{1}{x}\right)-y\left(1+\frac{1}{y}\right)$.
31. $a-[a-(1+a)]$.
32. $2p-3\{4-\overline{p-1}\}$.
33. $3x+y-\{4x-3(x-y)\}$.
34. $2[x-\overline{y-x}]-3(x+y)$.
35. $b-2\{c-4(b-c)\}$.
36. $a[a(a+b)-b(a-b)]$.

Fill in the gaps in the following :

37. $a-2b-2c=a-2(\quad)$.

38. $p-q+r=p-(\quad)$.

39. $x^2-xy=x(\quad)$.

40. $l+m-n=m-(\quad)$.

41. $2a-2c-3x-3z=2(\quad)-3(\quad)$.

42. $a^2+ab-p^2+pq=a(\quad)-p(\quad)=a(\quad)+p(\quad)$.

43. $x^2+xy-xz=x(\quad)=xy-x(\quad)$.

44. $3l+m-n=4l-(\quad)=3(l+m)-(\quad)$.

45. $1+x+x^2+x^3=1+x+x^2(\quad)=1+x^2+x(\quad)$.

46. Add $x+2(y-z)$ to $x-2(y+z)$.

47. Subtract $a-(b-c)$ from $c-(b-a)$.

48. Multiply $x-(x-y)$ by $y-(y-x)$.

49. What must be added to $p-(q+r)$ to give $p+(q+r)$?

50. What must be subtracted from $2(r-s)+t$ to leave $r-t$?

MISCELLANEOUS EXAMPLES

M. I.

1. A boy starts the term with $10x$ shillings. He spends x shillings at once and later on $3x$ shillings. How much has he then left ?

2. Express $12y$ shillings in £, and £$(1\frac{3}{4}z)$ in shillings.

3. A man is x years old. How old will he be in one year's time ? How old was he 3 years ago ?

4. Add together $2x^2$, $3x^2$ and $4x^2$. Subtract x^2 from the result.

5. A man cycles at $3x$ miles an hour. How far will he go in 3 hours ? How long will he take to go $10x$ miles ?

6. Evaluate $5x-2(2-x)$ if $x=1$. Simplify $5x-2(2-x)$ and evaluate your answer when $x=1$.

7. A walks a miles an hour ; B walks b miles an hour faster than A. How fast does B walk ? How far will B walk in 20 minutes ?

8. Subtract $a-3b$ from $2a-b+c$.

9. A tank contains $10x^2$ gallons of water. How many buckets, each containing $2x$ gallons, can be filled from it ?

10. A stick is a ft. b in. long ; what is the number of inches in the length of a stick twice as long ?

11. The length of a rectangle is $3x$ in. and its width is $2x$ in. What is (i) its perimeter, (ii) its area ?

12. Add together $7ab$ and ab ; subtract $5ab$ from the result.

13. Multiply x^4 by x^8 and divide the result by x^6.

14. Express z half-crowns (i) in £, (ii) in pence.

15. Which of the following are factors of $12xy$? $24xy$, $6x$, 12, $3x^2$, $12x^2y^2$, $4xyz$, xy, $60xy^2$, $2xy$. Which are common factors of $12xy$ and $12xz$?

16. Simplify, if possible, (i) $3x^2+4x^2$, (ii) $3x^2+4x$, (iii) $3x^2 \times 4x$.

17. A boy is now $2x$ years old, his brother is x years old. What is the sum of their ages (i) now, (ii) in 5 years' time ?

18. Write down the values of (i) $10+7$, (ii) $6 . 10+2$, (iii) $3 . 10+9$. How could the number $10x+y$ be pronounced, if x and y are whole numbers, not greater than 9 ?

19. Simplify (i) $50x^2yz \div 2x$, (ii) $(3x^3y^3)^2$, (iii) $\frac{2}{3} \times 2x$.

20. If tea costs xs. per lb. and coffee costs ys. per lb., write down the total cost of (i) 2 lb. of tea and 3 lb. of coffee, (ii) y lb. of tea and x lb. of coffee.

21. There are a boys in a school : at the end of term, b leave and c new boys come : how many are there in the school the next term ?

22. If $a=17$, $b=10$, $c=7$, find the value of (i) $a-b+c$, (ii) $a-\overline{b+c}$.

23. Find the H.C.F. of $8ayz$ and $10bxz$. Simplify $\frac{8ayz}{10bxz}$.

24. $10a+b$ people attend a meeting ; of these $5a$ get seats, how many have to stand ?

25. How many pence are there in £a bs. ?

26. In Fig. 71, the portion PQ of the rod AB is the same length as the portions AP and QB together. What is the length of AQ ?

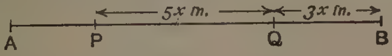


FIG. 71.

27. From $5a+b$ take $3a+c$.

28. Simplify (i) $\frac{xy}{axy}$, (ii) $\frac{x}{y} \times by$, (iii) $\frac{b}{c} \div \frac{b}{d}$.

29. On a bookshelf are $2x$ one-volume novels and $3x$ two-volume novels. How many volumes are there altogether ?

30. Simplify (i) $1-\frac{4}{9}$, (ii) $1-\frac{m}{n}$, (iii) $\frac{a}{b}-1$, (iv) $\frac{c}{d}\div 1$.

31. A bucket holds x lb. of water. A cistern when empty weighs y lb. and holds z buckets of water. What will be its weight when full ?

32. A grocer buys 24 lb. of butter ; $3x$ lb. of this prove to be bad. If I buy one-third of the good butter, how much do I buy ? If the price is bs. per lb., how much do I pay ?

33. In Fig. 72, how far is B from the mid-point of AC ?

FIG. 72.

34. Express by means of brackets, without simplifying, the following : From five times the excess of m over n subtract three times the sum of m and n ; subtract the result from $4m$.

35. If $x=13$, $y=3$, $z=1$, evaluate (i) $x-2y+z$, (ii) $x-2(y+z)$.

36. Fill in the blank space in

$$3\tfrac{1}{2}\times 17+2\tfrac{3}{4}\times 17-4\tfrac{1}{4}\times 17=(\quad)17.$$

Evaluate the expression.

37. Simplify (i) $\frac{4a}{11}+\frac{6a}{11}$; (ii) $\frac{4a}{11}+\frac{3a}{22}$; (iii) $\frac{5}{11a}-\frac{9}{22a}$.

38. Find the L.C.M. of $4p^2$ and $2pq$. Simplify $\frac{1}{4p^2}+\frac{1}{2pq}$.

39. A man buys p penny stamps and h halfpenny stamps ; how much change (in pence) will he receive from 10s. ?

40. Multiply $12a-b$ by $2ab$. What is the quotient if $24a^2b-2ab^2$ is divided by $12a-b$?

41. How many hours are there between $(n-2)$ o'clock p.m. and $(n+3)$ o'clock p.m. ?

42. Simplify (i) $\frac{b}{a}+\frac{c}{a}+\frac{d}{a}$; (ii) $\frac{p}{q}\times\frac{q}{r}$; (iii) $1-\frac{2m}{4m}$.

43. Simplify (i) $x(x^2-2x+1)+2(x-1)$,

(ii) $\frac{1}{a}(2a^2+ab)-\frac{1}{b}(ab+2b^2)$.

44. Divide $(2xy)^2-(2yz)^2$ by $2y$.

45. A is walking at $(x-1)$ miles an hour to Brighton and is 16 miles from Brighton. How far will he be from Brighton 2 hours later ?

46. An engine pumps x gallons of water per minute and a second engine pumps 50 gallons more than the first each minute. How much will the two together pump in an hour ?

47. Write down (i) the square of $5ab^3$, (ii) a square root of $9x^{16}$.

48. Simplify (i) $\frac{a^2}{bc}\times b$; (ii) $\frac{1}{a}-\frac{1}{ab}$; (iii) $\frac{x+y}{3}+\frac{x-y}{3}$.

49. Rewrite 1359×99 by putting $100-1$ for 99. Then evaluate the product.

50. (i) Find the H.C.F. of $12ax^2$ and $20\,axy$;

(ii) Simplify $\frac{12ax^2}{20axy}$; (iii) Simplify $(12ax^2+20axy)\div 4ax$.

51. Simplify (i) $4ab^2c\times 4a^2bc^2$; (ii) $\frac{8acy}{4y}$; (iii) $x\div\frac{1}{y}$.

52. Simplify (i) $\left(a\div\frac{1}{b}\right)+\left(b\div\frac{1}{a}\right)$; (ii) $\frac{2b^2}{c^2}-\left(\frac{ab}{c}\div\frac{ac}{b}\right)$.

53. Fill in the blanks in (i) $\frac{x}{y}=\frac{x^2}{\quad}$; (ii) $\frac{a}{x}=\frac{b}{\quad}\times\frac{\quad}{y}$.

54. Find (i) the perimeter, (ii) the area of Fig. 73, if all the corners are right-angled, the units being inches.

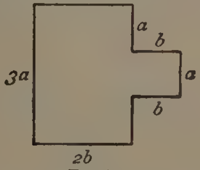


FIG. 73.

55. Simplify (i) $\frac{a}{x}-\frac{a-10}{x}$; (ii) $x-\frac{x}{100}$.

56. What is the excess of $a-b$ over $a-2b$? Check your answer for the values $a=10$, $b=3$.

57. Simplify $\frac{1}{2}(R+r)(R-r)$ if $R=5r$.

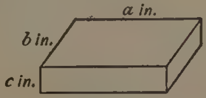


FIG. 74.

58. What is the sum of the areas of the faces of the rectangular block in Fig. 74. What does this become, in terms of c, if

$$c=\frac{b}{3}=\frac{a}{4}\ ?$$

59. Show that $(a+b)^2$ equals $a^2+2ab+b^2$, (i) when $a=6$, $b=4$; (ii) for any one pair of values you like to select.

60. Fill in the blanks in (i) $\frac{2}{3}=\frac{\quad}{100}$; (ii) $\frac{x}{y}=\frac{\quad}{z}$.

61 Fill in the blanks in (i) $\frac{a+b}{\quad}=\frac{1}{c}$; (ii) $\frac{1}{b+a}=\frac{\quad}{ac+bc}$.

62. How many hours are there between (i) x o'clock a.m. and y o'clock p.m. on the same day, (ii) x o'clock p.m. on Tuesday and y o'clock a.m. on Wednesday ?

63. Find, by inspection, the integral value of x if x^2+2 is a perfect cube. [There is only one answer.]

64. The year x A.D. contains $\frac{x}{6}$ days. What is x ?

CHAPTER III

GENERALISED ARITHMETIC.

Problems and Formulae.

Graphical Work. The graphical representation of statistics, etc. often forms part of the Arithmetic Course. For convenience, this has been placed in Chapter V.; but this work may be done concurrently with the earlier chapters, Chapters III. and IV. In the same way, Chapters VI. and VII. may be taken concurrently. **It is inadvisable to work through all the exercises, either in Chapter V. or Chapter VI., at any one time**; *it is suggested that the material of these chapters should be utilised at intervals during the other work of Parts I.-II.*

Numbers and Quantities. Letters in Algebra are used to represent *numbers*, **not** numbers-of-things. A letter may stand for 2, 3, 57, $\frac{3}{4}$, etc., but **not** for two-pence, three gallons, 57-days, $\frac{3}{4}$ mile, etc. A number-of-things is called a **quantity**; thus, x inches is a quantity, but x is a number. When dealing with quantities, always state what the unit is. Never say—let x be the cost of a house, or let t be the time taken for a journey—but say, let the house cost £x, let the journey take t hours or t minutes or t days, etc. Algebra is merely generalised Arithmetic; if you cannot see how to do a problem which contains letters instead of definite numbers, invent a similar problem with special numbers and work that out first; then apply the same argument to the letters. We may summarise these remarks as follows:

(i) **All letters used stand for numbers only, not quantities.**

(ii) **In every problem dealing with numbers-of-things, the units must always be stated.**

(iii) **If you cannot see how to work out a problem involving letters, invent a similar problem with numbers, and do that first.**

Exercise III. *a.* (Oral.)

Criticise any of the following statements that seem unsatisfactory, and suggest improvements.

1. If each book costs x, then two books cost $2x$.

2. If $2x$ is an even number, then $2x+2$ is also an even number.

3. If the length of a room is a feet and the width b feet, the area is ab.

4. The temperature was x and rose 3 degrees, so that it is now $x+3$.

5. Let the price of sugar be x.

6. Let the speed of a train be x.

7. Let the price of coffee per lb. be x.

8. If in the triangle ABC, $\angle ABC = x$ and $\angle ACB = y$, then $\angle BAC = 180 - x - y$.

9. Let the required even number be x.

10. If the number of inches in a side of a square is x, its area is x^2.

11. Let one book cost x and the other y; then, since one costs 3s. more than the other, $x - y = 3s.$

12. Let the distance from A to B be x and the speed from A to B be v feet per second.

13. In 1 mile, the speed of the train has increased x miles.

14. If the radius of a circle is x, its diameter is $2x$.

Unitary Method.

Example I. If a train is travelling at v miles an hour, how long will it take to go s miles?

Invent a similar question with numbers for letters. If a train is travelling at 40 miles an hour, how long will it take to go 57 miles?

It travels 40 miles in 1 hour;

$\therefore$ it travels 1 mile in $\frac{1}{40}$ hour;

$\therefore$ it travels 57 miles in $(\frac{1}{40} \times 57)$ hours $= \frac{57}{40}$ hours.

Now use the same method with letters.

It travels v miles in 1 hour;

$\therefore$ it travels 1 mile in $\frac{1}{v}$ hours;

$\therefore$ it travels s miles in $\left(\frac{1}{v} \times s\right)$ hours $= \frac{s}{v}$ hours.

Note. The *method* does not depend on the values of s and v, nor is it affected by whether s is greater than v or v greater than s.

For example, if a train travels $\frac{1}{2}$ mile in 1 minute,

$$\text{it travels 1 mile in } \frac{1}{\frac{1}{2}} \text{ minutes (}i.e.\text{ 2 minutes),}$$

$$\text{and it travels } \frac{3}{5} \text{ mile in } \left(\frac{1}{\frac{1}{2}} \times \frac{3}{5}\right) \text{ minutes.}$$

Areas and Volumes.

We shall assume the following formulae :

1. The area of a triangle of base b ft. and height h ft. is $\frac{1}{2}$bh sq. ft.
2. The circumference of a circle of radius r ft. is 2πr ft.
3. The area of a circle of radius r ft. is πr^2 sq. ft.
4. The volume of a solid of constant cross section A sq. ft. (e.g. a cylinder or prism) and length l ft. is Al cu. ft.

π stands for the number 3·14159... and is often taken to be 3·14 or $\frac{22}{7}$.

Note. Instead of ft., sq. ft., cu. ft., taken as units in the above formulae, any other corresponding units may be employed.

Example II. The volume of a glass tube is x c.c. and its length is y metres ; find its cross-section.

Its volume is x c.c. and its length is $100y$ cm.

$$\therefore \text{ the cross-section is } \frac{x}{100y} \text{ sq. cm.}$$

Example III. Fig. 75 represents a square plate from which four equal semicircles and one complete circle have been cut away, as shown ; the units are inches. Find the area of the surface of the plate and express it in the form, Area $= N \, . \, a^2$ sq. in., where N is a number correct to one place of decimals. Take $\pi = 3{\cdot}14$.

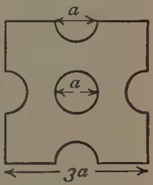


FIG. 75.

The area of the surface of the plate before the cutting is $\quad 3a \times 3a = 9a^2$ sq. in.

Four semicircles equal two complete circles ; $\therefore$ in all, 3 complete circles are cut away.

The diameter of each circle is a in. ; $\therefore$ radius of each circle is $\frac{a}{2}$ in.

$$\therefore \text{ area of each circle} = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4} \text{ sq. in.}$$

$$\therefore \text{ area of surface of plate} = 9a^2 - \frac{3\pi a^2}{4} = \left(9 - \frac{3\pi}{4}\right)a^2 \text{ sq. in.}$$

But

$$9-\frac{3\pi}{4}=9-\frac{3\times 3\cdot 14}{4}=9-\frac{9\cdot 42}{4}=9-2\cdot 4 \text{ to 1 place of decimals}$$
$$=6\cdot 6.$$

$\therefore$ area of surface of plate $=6\cdot 6a^2$ sq. in.

Note. The value of N is 6·6.

Exercise III. *b*.

1. If 1 gallon of water weighs 10 lb., write down the weight of (i) 3 gallons, (ii) $\frac{1}{4}$ gallon, (iii) x gallons, (iv) $\frac{1}{y}$ gallons.

2. If 10 yards of silk cost 30 shillings, write down the cost of (i) 1 yard, (ii) $\frac{1}{2}$ yard, (iii) b yards, (iv) $\frac{1}{c}$ yards.

3. If a yards of silk cost b shillings, find the cost of (i) 1 yard, (ii) x yards, (iii) $\frac{1}{2}$ yard, (iv) $2a$ yards, (v) $\frac{1}{2}a$ yards.

4. If a yards of silk cost b shillings, how much can be bought for (i) 1 shilling, (ii) x shillings, (iii) sixpence, (iv) $\frac{3}{4}b$ shillings ?

5. A clock loses t seconds an hour ; how many (i) seconds, (ii) minutes will it lose in 5 days ?

6. A man walks 3 miles an hour. How many minutes does he take to walk s miles ?

7. A man cycles 12 miles an hour. How far will he go in y minutes ?

8. £1 $=x$ francs, express 1 franc in £.

9. 1 oz. $=W$ gm., express 1 Kg. in lb.

10. A wall is 8 ft. high and l yd. long ; what is its area in sq. feet ?

11. A tile is p inches square ; how many are required per sq. foot ?

12. What is the cost in £ of a carpet a ft. long, b ft. wide at 5s. per sq. foot ?

13. The area of a door is A sq. ft. ; it is h ft. high ; what is its width ?

14. A vessel 20 cm. long, 12 cm. wide is filled with water to a depth of h mm. How many c.c. of water does it contain ?

15. If V cu. ft. of water are pumped into a tank l ft. long, b ft. wide, what will be the depth of water, (i) in ft., (ii) in inches?

16. The cross-section of the inside of a glass tube is A sq. mm. and it is a metre long. How many c.c. will it hold?

17. Fig. 76 represents a field: it is enclosed by a wooden fence h ft. high. Find the area of the fence in sq. ft.

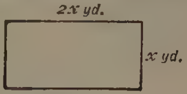


FIG. 76.

18. A garden path l yd. long, $1\frac{1}{2}$ yd. wide is covered with a layer of gravel d in. deep. How many loads of gravel are required, if a load contains 1 cu. yd.?

19. If a man pushes a mowing machine 10 in. wide a distance l yards, how many sq. ft. has he mown?

20. A 14-inch mowing machine must be pushed n times across a lawn in order to mow it; how many times will be required for a 10-inch machine?

21. What is (i) the circumference, (ii) the area of a circle of diameter d in.?

22. The minute hand of a clock is r ft. long. How far does its tip move in 20 minutes?

23. Fig. 77 represents a lawn, divided into two parts by a path 6 ft. wide, with two circular flower beds each of radius r ft. What is the area of the grass, in sq. ft.? [Take $\pi = \frac{22}{7}$.]

24. With the figure and data of No. 23, find the total length of the grass edging, in ft.

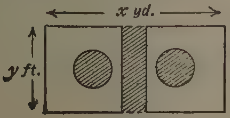


FIG. 77.

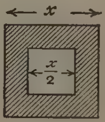


FIG. 78.

25. In a square tin plate of side x in., a square hole, side $\frac{x}{2}$ in., is pierced (see Fig. 78). Find the area of the surface and express it in the form $N \cdot x^2$ sq. in., where N is a number.

26. Find the shaded area in Fig. 79, the units being inches, and express it in the form $N \cdot a^2$ sq. in., where N is a number.

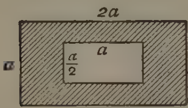


FIG. 79.

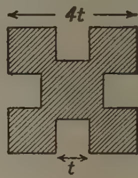


FIG. 80.

27. From a square plate of side $4t$ in., four squares, each of side t in., are cut away (see Fig. 80). Find the shaded area and express it in the form $N \cdot t^2$ sq. in., where N is a number.

28. Find the shaded area in Fig. 81 formed by cutting away a quadrant from a square and express it in the form $N \cdot x^2$ sq. in., where N is a number. [Take $\pi = \frac{22}{7}$.]

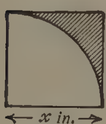


FIG. 81.


FIG. 82.

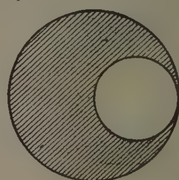


FIG. 83.

29. Find the shaded area in Fig. 82, enclosed by 4 quadrants, each of radius r in. Express it in the form $N \cdot r^2$ sq. in., where N is a number correct to 2 places of decimals. [Take $\pi = 3{\cdot}14$.]

30. Find the shaded area in Fig. 83 formed by cutting a circle of diameter d in. out of a circle of radius d in. Express it in the form $N \cdot d^2$ sq. in. where N is a number correct to one place of decimals. [Take $\pi = 3{\cdot}14$.]

31. Find the perimeter of the shaded area in (i) Fig. 78, (ii) Fig. 79, (iii) Fig. 80.

32. Find the perimeter of the shaded area corresponding to the data of No. 28, and express it in the form $k \cdot x$ in., where k is a number.

33. Find the perimeter of the shaded area corresponding to the data of No. 29.

34. Find the perimeter of the shaded area corresponding to the data of No. 30.

35. The shaded area in Fig. 78 is the base of a hollow brick of length $2x$ inches. Find the volume of the brick and express it in the form $c \,.\, x^3$, where c is a number.

36. Fig. 83 represents the section of a cylinder of length $4d$ inches, from which another cylinder has been cut away. With the data of No. 30, find the volume of the remaining solid and express it in the form $c \,.\, d^3$, where c is a number correct to one place of decimals.

37. (i) Find the shaded area in Fig. 84, the units being inches. If the area is $N \,.\, s^2$, what is N ?

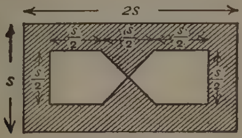


FIG. 84.

(ii) If Fig. 84 is the section of a metal plate of thickness $\frac{s}{20}$ inches, show that the volume of the plate is of the form $c \,.\, s^3$ cu. in. and find c.

(iii) If the metal in (ii) weighs 8 oz. per cu. in., find the weight of the plate.

EXTRA PRACTICE EXERCISES. E.P. 5.

SYMBOLICAL NOTATION.

1. How many inches are there in $2s$ feet, $\frac{1}{4}t$ yards ?

2. What is the length of a train which has n coaches, if each coach is l feet long and the engine is c feet long ?

3. A clerk writes n letters an hour, how long does he take to write p letters ?

4. A man smokes $\frac{1}{4}$ lb. of tobacco a week ; how long does W lb. of tobacco last him ?

5. A rug is 6 feet wide and 36 feet long ; what is its area in sq. ft., sq. yd. ?

6. The length of fence for a square enclosure is $2p$ yd.; what is the area enclosed?

7. A hall is twice as long as it is wide; it is x feet long; what is its floor-area?

8. In No. 7, the hall is 20 ft. high; what is the total area of the four walls?

9. A brick is 3 in. wide, 2 in. deep, t in. long; what is its volume?

10. In No. 9, the material of which the brick is made weighs W oz. per cu. in.; what is the weight of the brick?

11. How many tins measuring 2 in. by 3 in. by 4 in. can be filled from a tank containing V cu. ft. of water?

12. What volume of soil is removed when making a trench $2t$ ft. wide, $3t$ ft. deep, $9t$ ft. long? Answer in (i) cu. ft., (ii) cu. yd.

13. A pencil costs $\frac{3p}{2}$ pence; how many can I buy for half-a-crown?

14. I bicycle at the rate of $3s$ miles an hour, how far do I go in 40 minutes?

15. The average speed of a train is v miles an hour, how long does it take to go 50 miles? How long would it take if the speed was 10 miles an hour less?

16. Take the number N and from half of it subtract one-third of it.

17. Express in shillings the result of subtracting £x ys. from £y xs., where $y > x$.

18. From a square sheet of cardboard of side $3p$ in., a square of side $2p$ in. is cut away; what area remains?

19. A train has $11n$ compartments with 8 seats each and $2n$ compartments with 6 seats each; how many seats are there in the train?

20. In No. 19, find the number of such trains required to carry 2000 passengers, if each has a seat?

21. A brick is t in. broad, $2t$ in. long and h in. high. What is the sum of the length of its edges?

22. A handkerchief costs k pence and a pair of socks costs one shilling more than that. What is the total cost of 8 handkerchiefs and 4 pairs of socks, (i) in pence, (ii) in shillings ?

23. A tank is $3c$ ft. long and $2c$ ft. wide and contains c^3 cu. ft. of water. What is the depth of water, (i) in feet, (ii) in inches ?

24. A tank is $4p$ ft. long and $2p$ ft. wide and contains water to a depth of $3p$ ft. ; what is the area of the wetted surface ?

25. A bicycle wheel revolves N times in 100 yards ; how many times does it revolve in $\frac{1}{8}N$ miles ?

26. It takes x men t days to repair a certain road ; how long would it take $2x$ men ?

27. A photograph x in. long, y in. wide is printed on a sheet of paper $\frac{5x}{4}$ in. long, $2y$ in. wide. What area of the paper is not used ?

28. How many tiles measuring 8 in. by 6 in. are required for the floor of a hall b ft. broad, $2b$ ft. long ?

29. A certain milestone on the Winchester-Southampton road reads x miles to Winchester, $2x$ miles to Southampton. How far is Winchester from Southampton ? What is x if this distance is 12 miles ?

30. A shopkeeper would make a profit of £P by selling a table for £S. For what must he sell it to make a profit of (i) £$(2P)$, (ii) £p ?

31. In an examination, a girl, who scores n marks, fails by b marks ; a boy passes with c marks in hand. What did he score ?

32. The duty on a picture is two-fifths of its value. What is the value if the duty is £$(50P)$?

33. At a shooting gallery, you pay 2d. if you miss and receive 6d. if you hit the target. What is the result if you score x hits and y misses ?

34. A, B, C are three parcels ; A and B together weigh p lb., B and C together weigh q lb., A and C together weigh r lb. ; what is the total weight of A, B, C ?

35. A man starts from a boat-house B and rows downstream; he takes t minutes to row 4 miles; he then turns and takes $4t$ minutes to get back to B. How far from B is he at a time $2t$ minutes after leaving B?

Ratio and Percentage.

This section should be postponed, if the pupils have not done the corresponding work in Arithmetic.

Example IV. Divide £x into two shares (i) in ratio $2:3$, (ii) in ratio $a:b$.

For every £2 in one share, there is £3 in the other share, and this makes £$(2+3)$ = £5 in all.

$$\therefore \text{ the first share} = \tfrac{2}{5} \text{ of the whole} = \tfrac{2}{5} \text{ of } £x = £\frac{2x}{5};$$

and the second share $= \tfrac{3}{5}$ of the whole $= £\dfrac{3x}{5}$.

Similarly, for every £a in one share, there is £b in the other share, and this makes £$(a+b)$ in all.

$$\therefore \text{ the first share} = \frac{a}{a+b} \text{ of the whole} = \frac{a}{a+b} \text{ of } £x = £\frac{ax}{a+b};$$

and the second share $= \dfrac{b}{a+b}$ of the whole $= £\dfrac{bx}{a+b}$.

Example V. How much per cent. is x of $5x$?

$$\text{It is } \frac{x}{5x} \times 100 \text{ per cent.} = \tfrac{1}{5} \times 100 \text{ per cent.}$$
$$= 20 \text{ per cent.}$$

Example VI. A horse is sold for £a which cost £b; find the profit per cent.

Take a similar question in Arithmetic: a horse is sold for £37 which cost £32; find the profit per cent.

The profit is £$(37-32)$; the *cost* price is £32.

$$\therefore \text{ the profit is } \frac{£(37-32)}{£32} \times 100 \text{ per cent. of the cost price.}$$

Using letters, the profit is £$(a-b)$; the *cost* price is £b.

$$\therefore \text{ the profit is } \frac{£(a-b)}{£b} \times 100 \text{ per cent. of the cost price}$$
$$= \frac{a-b}{b} \times 100 \text{ per cent.} = \frac{100(a-b)}{b} \text{ per cent.}$$

EXERCISE III. c.

Simplify the following ratios :

1. (i) 6 shillings : £3 ; (ii) x shillings : £y.

2. (i) 4 half-crowns : 6 florins ; (ii) x half-crowns : y florins.

3. (i) 33 ft. per sec. : 30 miles per hour ;
(ii) u ft. per sec. : v miles per hour.

4. (i) £20 per month : £200 per year ;
(ii) £a per month : £b per year.

5. $x : (x + \frac{1}{2}x)$. 6. $(2x)^2 : (4x)^2$. 7. $(2x)^3 : (4x)^3$.

8. Two sums of money are in the ratio $a : b$; the first is £(ax), what is the second ? The first is £y, what is the second ?

9. (i) The price of a chair is raised from £4 to £6. What is the new price of a table, original price £9, if it is increased in the same ratio ?
(ii) The price of a chair is raised from £p to £q. What is the new price of a table, original price £x, if it is increased in the same ratio ?

10. A boy, who had x marks, received an additional y marks as a bonus. In what ratio has his total increased ? What bonus should be given to a boy who had b marks to preserve the same proportion ?

11. What is the ratio of the volumes of two cubes whose edges are b in., $2b$ in. respectively ?

12. Two fields are the same shape ; one is x yd. long and a yd. wide ; the other is y yd. long. What is its width ? What is the ratio of their areas ?

13. (i) If, in Fig. 85, the ratio $AB : BC = 3 : 7$, what is the ratio of AB to AC and of AC to BC ?

FIG. 85.

(ii) If in Fig. 85, the ratio $AB : BC = p : q$, what is the ratio of AB to AC and of AC to BC ?

14. If, in Fig. 85, $AB = 2x$ in., $BC = 4x$ in., $CD = 3x$ in., find the ratios (i) $AB : BC$; (ii) $BC : AD$; (iii) $AC : BD$.

15. If, in Fig. 85, $AB : BC : CD = p : q : r$, find the ratios

$$\text{(i) } \frac{AB}{AD}; \quad \text{(ii) } \frac{BC}{AD}; \quad \text{(iii) } \frac{AC}{BD}; \quad \text{(iv) } \frac{AD}{AC}.$$

16. The scale of a map is $1 : n$. Find in yards the length of a road represented on the map by a line, (i) 18 inches long, (ii) L inches long.

17. Express as a ratio (i) 20 per cent., (ii) R per cent.

18. What is 10 per cent. of (i) 80, (ii) A ?

19. What is the result of increasing by 20 per cent. (i) 60, (ii) N ?

20. How much per cent. is (i) 12 of 15, (ii) a of b ?

21. Express as a percentage (i) $\frac{3}{4}$, (ii) $\frac{a}{b}$.

22. What is R per cent. of £b ?

23. What number exceeds n by p per cent. ?

24. If r per cent. of a candle is burnt, what percentage remains ?

25. If x per cent. of a candle is burnt every hour, how long will the whole candle last ?

26. If n seeds are sown and if k per cent. of them come up, how many seeds are wasted ?

27. A man lends £P at 5 per cent. per annum, what is his yearly interest in shillings ?

28. A debtor pays p shillings in the £; what percentage of his debts does he pay ?

29. A mixture is made of C lb. of copper and T lb. of tin. How much per cent. of the mixture is copper ?

30. A pig cost £$(5x)$ and is sold at a profit of 30 per cent. Find the profit.

31. The price of tyres is raised 15 per cent.: the old price of a set was £N; what is the new price ?

32. Find the value in shillings of the sum of p per cent. of £10 and q per cent. of £5.

33. What is the simple interest on £100 for n years at r per cent. per annum ?

Formulae.

Some examples of the construction of formulae have already been given on p. 22, and their use has been illustrated on p. 32. Some more difficult applications are given below. For additional examples, see Exercise S. 1, p. 28.

Example VII. To find the law for Simple Interest.

The problem is to find the way in which the interest is connected with the sum of money lent, the time during which it is lent, and the rate per cent. at which interest is paid.

Suppose that £P is lent for T years at R per cent. per annum, and let the simple interest for that period be £I ? What is the connection between P, T, R, I ?

The interest on £100 for 1 year is £R ;

∴ the interest on £100 for T years is £$R \times T$;

∴ the interest on £1 for T years is £$\frac{R \times T}{100}$;

∴ the interest on £P for T years is £$\frac{P \times R \times T}{100}$.

But the interest is £I ;

$$\therefore \ \mathbf{I} = \frac{\mathbf{P} \times \mathbf{R} \times \mathbf{T}}{100}.$$

This is the required formula.

Exercise III. *d*.

1. A rectangular field is l yd. long, w yd. wide. Find a formula for its perimeter, P yd.

2. A rectangular block of stone is l ft. long, b ft. broad, h ft. high ; find a formula for (i) its volume, V cu. ft. ; (ii) its volume V' cu. yd.

3. Give a formula for finding the number of miles (m) that is equivalent to k km. Assume 8 km. = 5 mi.

4. Find a formula for the weight (W) in lb. of P pints of water. ["A pint of water weighs a pound and a quarter."]

5. The area of a rectangle is A sq. in. and one side is l in., find the length of the other side and a formula for the perimeter, P in.

6. A man starts with a salary of £P a year, and each year his salary is increased by £M a year. Find a formula for the sum of money £S he receives for his nth year of service.

7. A room is l ft. long, b ft. broad, h ft. high ; find a formula for the total area, A sq. ft., of its walls.

8. Find a formula for the number N which exceeds the number n by R per cent.

9. Assuming that 12 o'clock is the middle of the day, find a formula for the length, l hours, of the day when the sun rises at x a.m.

10. In Fig. 86, AE bisects $\angle BAC$; find a formula for $\angle DAE$ ($x°$) in terms of $\angle DAB$ ($y°$).

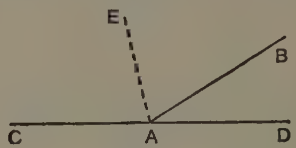


FIG. 86.

11. In Fig. 87, find a formula for x in terms of a, b, c.

FIG. 87.

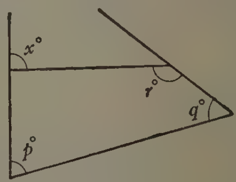


FIG. 88.

12. In Fig. 88, find a formula for x in terms of p, q, r.

13. When the shadow of a telegraph pole h ft. high is l ft. long, the shadow of a tower is m ft. long. Find a formula for the height, x feet, of the tower.

14. A beaker when empty weighs a gr. ; when full of water, it weighs b gr. ; when full of methylated spirit, it weighs c gr. Find a formula for the specific gravity of methylated spirit, *i.e.* the ratio of the weight of any amount of methylated spirit to the weight of the same amount of water.

15. An n-sided figure has $\frac{1}{2}n(n-3)$ diagonals. Verify this formula for a hexagon (6 sides). Find by inspection the number of sides of a polygon, which has 27 diagonals.

SUPPLEMENTARY EXERCISE. S. 4

1. If 20 lb. of coffee cost £1 15s., how much coffee can be bought for £P ?

2. If 30 lb. of tea cost £3 10s., how much tea can be bought for £$x+y$s. ?

3. A town has provisions for N people for t days. How long should these provisions last, if there were n people ?

4. Find the area of the end wall of the building, shown in Fig. 89.

Express this in terms of x, if $z=3x$ and $y=\dfrac{5x}{4}$.

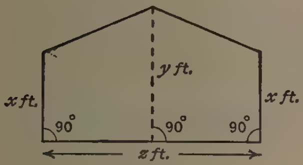


FIG. 89.

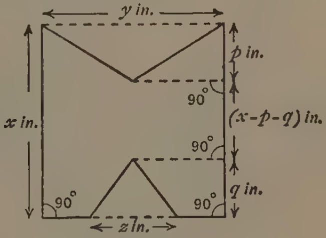


FIG. 90.

5. Find the area of Fig. 90.

6. A salesman deducts $5p$ per cent. of the marked price, for cash. How many shillings in the £ is this ?

7. How many coins, each weighing $\frac{2}{5}$ oz., can be made from W oz. of metal ?

8. The thickness of a sheet of paper is $\dfrac{2}{n}$ of an inch. How many sheets are there in a pile of paper, h in. high ?

9. On a railway the telegraph poles are placed d feet apart. A train passes n poles per minute. Find a formula for its speed, v miles per hour.

10. Find the volume of the circular cylinder in Fig. 91.

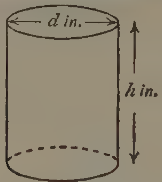


FIG. 91.

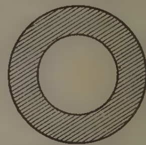


FIG. 92.

11. Fig. 92 represents the cross-section of a glass tube, internal radius r in., external radius R in.; the tube is l in. long. How many cu. in. of glass are there?

12. How many times can a cylindrical glass, radius r in., height h in., be filled from a cylindrical jug, radius R in., height H in.?

13. A cu. ft. of copper is drawn out into a wire, of cross-section $\frac{1}{s}$ sq. in.; what is the length in yards of wire obtained?

14. If I buy a War Savings Certificate for 16s. now, I shall receive either £1 in 5 years' time or 25s. in 10 years' time. If I spend £P now on certificates, what shall I receive in 5 years' time? If I wish to receive £A in 10 years' time, how many certificates must I buy now?

15. The wheel of a car is d ft. in diameter and is making r revolutions per second. Find a formula for the speed, v miles an hour, of the car.

16. Fig. 93 represents a rectangular plate from which two semicircles, each of radius r in., have been removed. Find (i) its area, (ii) its perimeter.

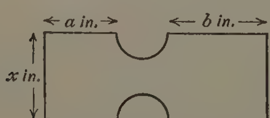


FIG. 93.

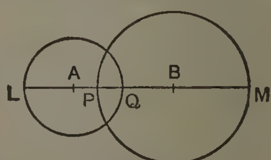


FIG. 94.

17. In Fig. 94, A, B are the centres of two circles of radii a, b cm.; the centres are d cm. apart. Find the lengths of LM and PQ.

18. Instantaneous photographs are taken of a revolving circular wheel for a cinematograph, 20 snap-shots being taken each second. The wheel has six equal and equally spaced spokes and is making n revolutions per second. Find the condition that the wheel should appear on the film to be at rest.

EASY REVISION PAPERS. A. 1-5

A. 1

1. If $x=5$, $y=1$, find the values of (i) $3x$; (ii) xy^2 ; (iii) $2x-6y$.

2. A boy was 10 years old x years ago ; how old will he be in y years' time ? When will he be z years old ?

3. Write down (i) the square of $4a^3$; (ii) the value of $2x^5 \div x^3$; (iii) the value of $4x^2y \div 2xy^2$.

4. Simplify (i) $\frac{x}{3}+\frac{2x}{3}$; (ii) $a^2b^2 \div \frac{a^2}{b^2}$; (iii) $1-\frac{p}{q}$.

5. Give a general statement which includes the following: $3^2+2\times 3+1$ is a perfect square, so also are $4^2+2\times 4+1$, $5^2+2\times 5+1$, $6^2+2\times 6+1$.

A. 2

1. If $x=6$, find the values of (i) $\frac{2x}{15}$; (ii) $\frac{1}{2}x^2$; (iii) x^2-5x.

2. Fig 95 represents two people leaving A and B at the same time. How far apart are they after (i) 1 hour, (ii) half an hour ? What can you say about u, v if they meet after 75 minutes ?

FIG. 95.

3. Find the H.C.F. and L.C.M. of $3x^2$ and $6x^2y$. Simplify

$$\text{(i) } \frac{3x^2}{6x^2y}; \quad \text{(ii) } \frac{1}{3x^2}+\frac{z}{6x^2y}.$$

4. Simplify $a(a-2b)+b(2a-b)$.
Subtract $a+b-c$ from $a-b+c$.

5. x versts equals $\left(x \text{ miles} - \frac{x}{3} \text{ miles}\right)$. Express 1 mile in versts. Also use the given statement to express 42 versts in miles.

A. 3

1. By drawing a figure as in Ex. II. c, find an expression for $(t+2)(t+3)$ which does not contain brackets (see p. 45).

2. It costs $\frac{x}{6}$ shillings to telegraph x words to France. How many words can be sent for half-a-crown ?

3. Simplify (i) $\frac{2}{3}+\frac{3}{2}$; (ii) $\frac{a}{b}+\frac{b}{a}$; (iii) $\frac{1}{a}+\frac{1}{b}$.

4. Find the H.C.F. and L.C.M. of $6a^2bc$ and $8ab^2c$. Show that the product of the H.C.F. and L.C.M. is equal to the product of the given expressions.

5. Express by means of brackets the amount by which the product of $a-b$ and c is greater than the product of $a-c$ and b; then simplify it.

A. 4

1. In Feb. 1917 the Food Controller allowed each person 4 lb. of bread a week. How many lb. would a family of n people obtain for 5 weeks ?

2. If $x=3$, find the value of

$$\text{(i) } x(x-1); \quad \text{(ii) } 1-\frac{x}{x+1}; \quad \text{(iii) } \sqrt{(2x^2-2)}.$$

3. State in words the meaning of $x-(y+z)$. Simplify

$$\{x-(y-z)\}-\{x-(y+z)\}.$$

4. Add together x^3-x^2+1, x^2-x+1, x^3-x+1.

5. My yearly bill for *The Times* was x shillings when each copy cost $1\frac{1}{2}$d.; what was it when the price was raised to 2*d.* ? How much extra has to be paid each year ?

A. 5

1. For a book published at P shillings, the cash price is $\left(P-\frac{P}{4}\right)$ shillings. What is the cash price of a book published at 24s. ? What is the published price of a book whose cash price is C shillings ?

2. (i) Multiply $2ab$ by $3ac$ and divide the result by abc.

(ii) Multiply $(a-b)$ by 3 and subtract the result from 4 times $(a+b)$.

3. Simplify

(i) a^2+3a^2; (ii) a^2+3a-a; (iii) $a^2 \div \frac{1}{2}a$.

4. Find in terms of x the third angle of the triangle in Fig. 96. Show that the triangle is isosceles if $x=10$ or 34 or 40.

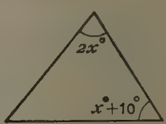


FIG. 96.

5. A man plants 5 potatoes to every sq. yard, and 8 potatoes weigh 1 lb. How many lb. of potatoes must he have, for planting N sq. yd. ?

CHAPTER IV

PROBLEMS AND SIMPLE EQUATIONS.

Example I. Express by an equation the following statement: A number is such that when it is added to 13, the result is 21.

We therefore say

$$13 + \text{the required number} = 21.$$

But it saves time to use a letter to represent the required number or "*the unknown*," as it is usually called.

Let n represent the required number, then

$$13 + n = 21.$$

A statement of this kind, which consists in saying that one expression equals another expression, is called an **equation**; the process of discovering the unknown number is called **solving the equation**; and the value of the unknown number is called the **root of the equation.**

Example II. A box and its contents are together worth thirty shillings; the contents are worth four times as much as the box. Express these facts by an equation.

The value of the box + four times the value of the box = thirty shillings.

To put this in a short-hand form, we say

Let the value of the box be v shillings.

Then $\quad v$ shillings $+ 4v$ shillings $= 30$ shillings.

$$\therefore\ v + 4v = 30.$$

Note. Always state clearly what the unknown letter represents. Remember that a letter represents a number, not *a number-of-things.*

Exercise IV. *a.*

In Nos. 1-14, write the given statement in the form of an equation. You need not try to discover the unknown number, but *keep the equation for future reference* (see p. 104).

1. I think of a number, then subtract 15; the result is 8.
2. I think of a number, then double it; the result is 38.

3. I think of a number, then divide it by 3, then add 5; the result is 20.

4. I think of a number, then add 12; the result is the same as multiplying the original number by three.

5. From the cube of a number, I subtract the square of the same number, the result is 180.

6. The sum of two consecutive integers is 37.

7. The average of 25, 40 and another number is 37.

8. The difference between a number and its square is 72.

9. To a certain number, I add one-half of that number and obtain 81.

10. The denominator of a fraction is greater than the numerator by 16 and the fraction equals 0·6.

11. One number is double another and their product is 288.

12. I think of a number, add 5, double the result, subtract the original number and finally obtain 17.

13. If a certain number is increased by 25 per cent., the result is 200.

14. The sum of a number and its cube is greater than its square by 21.

From each of the statements in Nos. 15-32, obtain (i) an equation connecting quantities, (ii) an equation connecting numbers. In each case, *state clearly what the unknown represents.* You need not try to discover the unknown, but *keep the equation for future reference* (see p. 104).

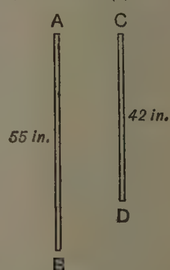


FIG. 97.

15. A box and its contents weigh 48 lb.; the contents weigh three times as much as the box.

16. The rods AB and CD in Fig. 97 are cut down by equal amounts, so that one is just twice the other.

17. A is now 39 years old and B is 15; in a certain number of years' time, A will be just twice B's age.

18. In Fig. 98, $\angle BAC = 3\angle ABC = 3\angle ACB$.

19. In Fig. 98, BP exceeds PC by $1\frac{1}{2}$ in., and $BC = 5$ in.

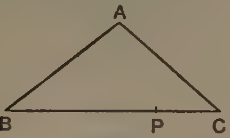


FIG. 98.

20. The perimeter of a rectangle is $5\frac{1}{2}$ ft. and the length exceeds the breadth by 3 inches.

21. A hall is twice as broad and three times as long as it is high: it contains 6000 cu. ft. of air.

22. A certain number of half-crowns and two more than that number of florins are worth altogether £2.

23. A man buys a certain number of oranges at a penny each and one-third of that number for twopence each: his bill is half a crown.

24. The perimeter of a semicircular area of a certain radius is $13\frac{1}{2}$ inches.

25. The boundary of the shaded area in Fig. 99, formed by cutting a quadrant away from a square, is 5 inches.

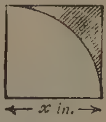


FIG. 99.

26. The shaded area in Fig. 99 is $10\frac{1}{2}$ sq. cm.

27. When the price of tea has been raised 10 per cent., it costs 2s. 9d. per lb.

28. A certain number of pounds of tea worth 2s. 3d. per lb. is mixed with 20 lb. of tea worth 1s. 9d. per lb.; the mixture is worth 1s. 11d. per lb.

29. A man walks at 4 miles an hour from his house to a station and returns home at 3 miles per hour; both journeys together take $3\frac{1}{2}$ hours.

30. One tap pours water into a bath twice as fast as another; together they take 6 minutes to fill the bath, which holds 90 gallons.

31. The sum of the angles of an n-sided polygon is 14 right angles. [Use the fact stated in Ex. IV. *f*. No. 27.]

32. A certain sum of money lent for 6 months at 8 per cent. amounts to £78.

Write down statements about *numbers* which can be expressed by the equations in Nos. 33-38.

33. $n-7=19$.

34. $n+3=\frac{1}{4}n^2$.

35. $N-\dfrac{N}{3}=30$.

36. $(n-1)+n+(n+1)=42$.

37. $N-40=90-N$.

83. $n+\dfrac{1}{n}=2\frac{2}{3}$.

Write down statements about *quantities* which can be expressed by the equations in Nos. 39-45; the kind of quantity is shown by the words in brackets.

39. $W+4W=50$ [Weight].

40. $x+x+x=180$ [Angle].

41. $t\times 3t\times 4t=144$ [Volume].

42. $b(b+2)=48$ [Area].

43. $C+\dfrac{4C}{100}=78$ [Percentage].

44. $\frac{1}{2}x+2\frac{1}{2}x=1\frac{1}{2}\times 20$ [Money].

45. $\dfrac{d}{3\frac{1}{2}}+\dfrac{d}{4}=3\frac{3}{4}$ [Distance and Time].

Solving Equations.

Example III. What is x, if $x+6=19$?

I think of a number; then add 6; the result is 19. What number did I choose?

By adding 6, I obtained 19; therefore the number I chose was

$$19-6=13;$$

$$\therefore\ \mathbf{x=13}.$$

Check: $x+6=13+6=19$.

Example IV. What is x, if $7x=63$?

I think of a number; then multiply it by 7; the result is 63. What number did I choose?

$\frac{63}{7}\times 7=63$; therefore the number I chose was $\frac{63}{7}=9$;

$$\therefore\ \mathbf{x=9}.$$

Check: $7x=7\times 9=63$.

Exercise IV. *b*.

Express each equation as a "think of a number" puzzle; then find the answer and check it.

1. $n+8=17$.
2. $n-5=11$.
3. $3y=42$.
4. $\frac{z}{2}=13$.
5. $x+1\frac{1}{2}=2$.
6. $x-3\frac{1}{2}=5$.
7. $24=4y$.
8. $x-7=0$.
9. $10x=0$.
10. $\frac{x}{7}=8$.
11. $2y=2\frac{1}{2}$.
12. $7-z=2$.
13. $\frac{n+2}{3}=7$.
14. $\frac{N}{3}+2=7$.
15. $\frac{x}{5}-1=3$.
16. $7=2+\frac{y}{2}$.
17. $0=11-y$.
18. $2(x+3)+5=23$.
19. $n+2n=18$.
20. $3z=1\frac{1}{2}$.
21. $\frac{x}{4}-3=8$.
22. $15-y=4$.
23. $1+2N=19$.
24. $\frac{z}{3}+1=6$.
25. $4x-x=21$.
26. $\frac{y-1}{2}=8$.
27. $2N=N+10$.
28. $\frac{n}{4}-2=3$.
29. $9-z=0$.
30. $\frac{3}{4}x=9$.

In the last Exercise, the value of the unknown number has been found by commonsense: we can save time and trouble by showing that the procedure always involves one or more of four simple notions.

Example V. Solve $n-17=46$.

Since the number $n-17$ and 46 are equal, if we add 17 to each of them, the results will be equal.

$$\therefore\ n-17+17=46+17;$$

$$\therefore\ n=63.$$

Example VI. Solve $x+17=46$.

Since the numbers $x+17$ and 46 are equal, if we subtract 17 from each of them, the results will be equal.

$$\therefore\ x+17-17=46-17;$$

$$\therefore\ x=29.$$

Example VII. Solve $7y=91$.

Since the numbers $7y$ and 91 are equal, if we divide each number by 7, the results will be equal.

$$\therefore\ 7y \div 7 = 91 \div 7;$$
$$\therefore\ y = 13.$$

Example VIII. Solve $\frac{z}{3}=14$.

Since the numbers $\frac{z}{3}$ and 14 are equal, if we multiply each number by 3, the results will be equal.

$$\therefore\ \frac{z}{3} \times 3 = 14 \times 3;$$
$$\therefore\ z = 42.$$

The solution of every simple equation is performed by applying one or more of these four arguments.

Example IX. Solve $5x-11=2x+25$.

Add 11 to each side.

$$\therefore\ 5x - 11 + 11 = 2x + 25 + 11$$
$$\text{or}\ \ 5x = 2x + 36.$$

Subtract $2x$ from each side.

$$\therefore\ 5x - 2x = 2x + 36 - 2x$$
$$\text{or}\ \ 3x = 36.$$

Divide each side by 3.

$$\therefore\ 3x \div 3 = 36 \div 3;$$
$$\therefore\ x = 12.$$

To make sure that this answer is correct, we check the results as follows:

Take the given equation.

$$\text{Left side} = 5x - 11 = 5 \times 12 - 11 = 60 - 11 = 49.$$
$$\text{Right side} = 2x + 25 = 2 \times 12 + 25 = 24 + 25 = 49.$$
$$\therefore\ \text{when } x = 12, \text{ left side} = \text{right side}.$$

Example X. Solve $\frac{x}{3}-2=\frac{3x}{4}-\frac{x}{2}$.

Multiply each side by the L.C.M. of the denominators; in this case the L.C.M. of 3, 4, 2 is 12.

$$\therefore\ \frac{12x}{3} - 24 = \frac{36x}{4} - \frac{12x}{2};$$
$$\therefore\ 4x - 24 = 9x - 6x;$$
$$\therefore\ 4x - 24 = 3x.$$

Add 24 to each side. $\therefore\ 4x = 3x + 24.$

Subtract $3x$ from each side.

$$\therefore\ 4x - 3x = 24;$$

$$\therefore\ x = 24.$$

Check: Left side $= \frac{x}{3} - 2 = \frac{24}{3} - 2 = 8 - 2 = 6.$

Right side $= \frac{3x}{4} - \frac{x}{2} = \frac{3 \times 24}{4} - \frac{24}{2} = 18 - 12 = 6;$

$\therefore$ when $x = 24$, left side = right side.

When checking an equation

(1) Substitute for "the unknown" in each side *separately*, as shown above: *never* simplify the two sides together.

(2) Always substitute in the equation, as it is given, *not* in any simplified form of it. The object of checking is to make sure your answer is right. You cannot be certain of this unless you substitute in the actual equation given you.

When solving an equation

(1) Always start by copying down the equation you have to solve *exactly as it stands in the book*: do not try to simplify it in your head.

(2) Never write your work in the following way:

$$2x = 8$$

$$= \ x = 4.$$

The symbol = is used to show that two expressions are equal; it does not mean that one statement follows from another statement.

The *correct* method of writing down the argument is as follows:

$$2x = 8;$$

$$\therefore\ x = 4.$$

The four arguments used in solving simple equations may be summarised as follows:

(i) *Equal numbers may be added to each side.*

(ii) *Equal numbers may be subtracted from each side.*

(iii) *Each side may be multiplied by equal numbers.*

(iv) *Each side may be divided by equal numbers.*

From (i), if $x - a = b$, then $x = b + a$.

From (ii), if $x + a = b$, then $x = b - a$.

From these results, we see that **we may move any term from either side of an equation to the other side if we change the sign in front of it.**

Note. It is sometimes convenient to reverse the order of an equation. Thus if $3 = x$, we can at once say $x = 3$, without using any of the arguments (i)-(iv) above.

Example XI. Solve $\frac{5}{x}=\frac{2}{3}$.

Multiply each side by $3x$, $\therefore\ 15=2x$;

$\therefore\ 2x=15$.

Divide each side by 2, $\therefore\ x=\frac{15}{2}=7\frac{1}{2}$.

At first, the reason for each step in the solution of an equation should be stated in words as is done in Examples IX., X. But eventually the process becomes mechanical.

If we are given that $\frac{a}{b}=\frac{c}{d}$ and we multiply each side by bd, we obtain $ad=bc$. This process is called **cross-multiplying.** Thus, if the given equation is $\frac{x-3}{x-5}=\frac{6}{7}$, we obtain, by cross-multiplying, $7(x-3)=6(x-5)$. But this form of statement should not be used in the early stages.

EXERCISE IV. *c*.

Solve the equations in Nos. 1-10, explaining each step in the argument as in Examples IX., X. above: and check each answer.

1. (i) $3n=21$; (ii) $8x=64$;
   (iii) $1\frac{1}{2}R=12$; (iv) $0{\cdot}3m=1{\cdot}2$.

2. (i) $a-4=7$; (ii) $p-7=0$;
   (iii) $z-2\frac{1}{2}=6\frac{1}{2}$; (iv) $w-3{\cdot}2=1{\cdot}9$.

3. (i) $l+5=12$; (ii) $x+2\frac{1}{2}=7$;
   (iii) $t+\frac{2}{5}=1$; (iv) $t+7{\cdot}4=9$.

4. (i) $\frac{1}{8}p=3$; (ii) $\frac{y}{4}=7$;
   (iii) $\frac{n}{3}=\frac{2}{7}$; (iv) $\frac{q}{5}=4{\cdot}2$.

5. (i) $\frac{4R}{5}=2$; (ii) $\frac{3p}{8}=12$;
   (iii) $\frac{2m}{3}=10$; (iv) $\frac{3x}{5}=1$.

6. (i) $3=n-2$; (ii) $7=p+5$;
   (iii) $4\frac{1}{2}=t+1\frac{1}{4}$; (iv) $6{\cdot}2=3{\cdot}8+k$.

7. (i) $3y-4=8$; (ii) $4R+3=13$;
(iii) $5N+2=17$; (iv) $7z-3=25$.

8. (i) $6y-15=0$; (ii) $5y+2=7\frac{1}{2}$;
(iii) $7l-3=9$; (iv) $9t+8=20$.

9. (i) $0{\cdot}3x=6$; (ii) $2{\cdot}5t=11$;
(iii) $1{\cdot}6z=12$; (iv) $0{\cdot}5k=0$.

10. (i) $\frac{3y}{2}=1$; (ii) $\frac{4a}{7}=\frac{2}{5}$;
(iii) $\frac{3t}{8}=0$; (iv) $\frac{5t}{8}=\frac{1}{5}$.

Solve the following equations and check each answer.

11. $2p-8=p-3$. **12.** $2l+4=19-l$. **13.** $t+7=17-4t$.

14. $3(n-7)=12$. **15.** $4(2k+1)=20$. **16.** $7(3y-1)=28$.

17. $4(t-5)=0$. **18.** $l-\frac{1}{3}l=6$. **19.** $m+\frac{m}{5}=24$.

20. $5=3R$. **21.** $0=2t-7$. **22.** $10y=y$.

23. $x-\frac{2x}{7}=10$. **24.** $\frac{p}{2}-\frac{p}{3}=1$. **25.** $\frac{l}{3}=1+\frac{l}{4}$.

26. $\frac{R}{5}-\frac{2}{7}=0$. **27.** $\frac{1}{2}(3x-1)=7$. **28.** $3=\frac{1}{4}(2x+1)$.

29. $3t=5{\cdot}7$. **30.** $\frac{1}{4}k=1{\cdot}7$. **31.** $2{\cdot}4y=6$.

32. $\frac{3}{x}=\frac{2}{5}$. **33.** $\frac{3}{4}=\frac{2}{z}$. **34.** $\frac{5}{2p}=\frac{1}{6}$.

35. $\frac{2R}{3}-\frac{R}{2}=1$. **36.** $\frac{t}{3}+\frac{t}{5}=0$. **37.** $\frac{2y-3}{5}=3$.

38. $\frac{6x}{7}+2=11$. **39.** $5=\frac{3k-1}{4}$. **40.** $2{\cdot}7=\frac{3}{4}m$.

41. $\frac{t+1}{t+3}=\frac{5}{6}$. **42.** $\frac{y-1}{3}=\frac{2y+1}{7}$. **43.** $\frac{2R+4}{5}=\frac{4-R}{3}$.

44. $\frac{3x}{5}-\frac{x}{2}=\frac{1}{2}$. **45.** $1-\frac{7a}{2}=a-5$. **46.** $\frac{1}{z}+\frac{1}{3z}=\frac{1}{12}$.

47. $R-0{\cdot}7R=12$. **48.** $\frac{x}{0{\cdot}3}=4$. **49.** $\frac{y}{3}\div 1\frac{1}{2}=1\frac{1}{3}$.

EXTRA PRACTICE EXERCISES. E.P. 6.

SIMPLE EQUATIONS.

Solve the following equations and check each answer.

1. $7n = 28$. 2. $5n = n + 8$. 3. $3N - 5 = 7$.
4. $12 = 3N$. 5. $35 = 5x$. 6. $x - 6 = 0$.
7. $7x - x = 18$. 8. $\frac{t}{3} = 10$. 9. $4y = 1\frac{1}{2}$.
10. $\frac{2h}{3} = 14$. 11. $\frac{12}{n} = \frac{3}{4}$. 12. $x - \frac{x}{2} = 36$.
13. $10t - 1 = 3t + 13$. 14. $8x - 20 = 5x + 7$.
15. $z - 7 = 10 - z$. 16. $\frac{n}{2} + \frac{n}{3} = 20$.
17. $0 = 3x - 12$. 18. $5x = 3x$. 19. $A + \frac{A}{5} = 18$.
20. $11p - 5 - 4p = 37$. 21. $3(R - 2) = 27$.
22. $3x = x + 16$. 23. $2x - 7 = 0$. 24. $25 - x = 4x$.
25. $2(x - 1) = 14$. 26. $12 = 7x$. 27. $9(x - 2) = 7$.
28. $3x - 1 = 0$. 29. $7x = 4x$. 30. $5x = 7\frac{1}{2}$.
31. $\frac{3x}{4} = 2$. 32. $\frac{x}{3} = \frac{7}{9}$. 33. $\frac{5}{2x} = \frac{3}{4}$.
34. $2x = 3 \cdot 8$. 35. $\frac{1}{2}x = 0 \cdot 7$. 36. $\frac{x}{8} \div \frac{3}{2} = \frac{5}{6}$.
37. $x - \frac{3x}{4} = 7$. 38. $2x + \frac{2x}{3} = 40$. 39. $\frac{x}{2} - \frac{x}{5} = 6$.
40. $\frac{1}{6}x - \frac{1}{2} = 0$. 41. $\frac{4x}{7} - 2 = 18$. 42. $x + \frac{1}{6}x = 5\frac{1}{4}$.
43. $\frac{x}{2} + \frac{x}{3} = 0$. 44. $\frac{1}{2}(x - 1) = 6$. 45. $\frac{2}{3}(x + 5) = 16$.
46. $\frac{1}{4}(2x + 1) = 2\frac{1}{4}$. 47. $3 = \frac{1}{5}(x - 2)$. 48. $1 = \frac{1}{4}(2x + 1)$.
49. $\frac{x - 1}{4} = 6$. 50. $\frac{2x + 10}{3} = 16$. 51. $\frac{3x - 1}{5} = 2\frac{1}{5}$.
52. $\frac{x - 2}{x - 1} = \frac{3}{4}$. 53. $3 = \frac{x - 2}{5}$. 54. $7 = \frac{2x + 5}{3}$.
55. $3x = 5 \cdot 7$ 56. $\frac{1}{2}x = 0 \cdot 7$. 57. $1 \cdot 2x = 6$.
58. $0 \cdot 1x = 2 \cdot 7$. 59. $3 \cdot 5x = 5 \cdot 6$. 60. $x - 0 \cdot 3x = 14$.

Exercise IV. *d*.

Solve the following equations :

1. $15x+10=5x+90.$
2. $4x-25+3x-10=0.$
3. $3(2x-3)=x+1.$
4. $4-4(1-x)=36.$
5. $10(x+4)-7(x-3)=100.$
6. $2(5x-1)-3(2+x)=6.$
7. $10(2+x)-51=3(x-1)-2(x+5).$
8. $5(5x-11)=6(2x-7).$
9. $2(x-2)+3(x-3)+4(x-4)=0.$
10. $78=8x-3(7-x).$
11. $\frac{2x}{3}-\frac{3}{2}=\frac{x}{4}+1.$
12. $\frac{2x}{3}+\frac{3x}{5}=5-\frac{2x}{5}.$
13. $1{\cdot}7x+1{\cdot}5=0{\cdot}6x+7.$
14. $\frac{2}{3}(x-3)=\frac{1}{2}(x-2).$
15. $x-\frac{2}{5}(2x-3)=3.$
16. $1=3-\frac{1}{7}(3x-1).$
17. $x=4-\frac{2}{3}(x+1).$
18. $\frac{1}{3}(x-1)-\frac{1}{4}(x+5)=0.$
19. $2x-\frac{x-6}{4}=9.$
20. $2+\frac{x-2}{3}=5-\frac{5x+1}{6}.$
21. $0=\frac{3-4x}{4}-\frac{3(x-1)}{2}.$
22. $2-\frac{1-x}{6}=3-\frac{5-x}{8}.$
23. $n-(2n-1)=5-2(n-1).$
24. $y=2(90-y).$
25. $r+\frac{22r}{7}=10.$
26. $P+\frac{5P}{100}=63.$
27. $Q-\frac{30Q}{100}=35.$
28. $\frac{17}{2}\{2+(17-1)d\}=221.$
29. $42=20\times 3-\frac{1}{2}f\,.\,3^2.$
30. $\frac{3}{4}(b+4)=\frac{2}{3}(b+5).$
31. $x^2+1=x(x+3)-5(x-7).$
32. $n^2-1-n(n-1)=3.$
33. $\frac{5}{x}+\frac{3}{7}=\frac{41}{14}.$
34. $\frac{2}{y}+\frac{3}{5y}=\frac{1}{5}.$
35. If $A=lb$, and $A=14$, $b=4$, find l.
36. If $A=\frac{1}{2}bh$ and $A=15$, $b=10$, find h.
37. If $A=\frac{1}{2}(a+b)h$ and $A=30$, $a=8$, $b=6$, find h.
38. If $s=\frac{1}{2}ft^2$ and $s=24$, $t=4$, find f.
39. If $f=\frac{M-m}{M+m}\cdot g$ and $f=4$, $g=32$, $M=9$, find m.
40. If $s=\frac{44r}{7}(h+r)$ and $r=3$, $s=100$, find h.
41. If $\frac{x}{2}-1$ is twice as large as $\frac{x}{3}-4$, find the value of each.

Easy Problems.

In Exercise IV. *a*, various statements have been expressed in the form of equations, and in Exercises IV. *c*, *d* certain equations have been solved. In order to obtain the answer to a problem, it is usually necessary to combine these two processes.

Exercise IV. *e*.

Take each of the equations you wrote down in Exercise IV. *a*. Nos. 1-32, and try to solve it and so obtain the answer to the problem described.

Make a list of any equations you obtained, which you are still unable to solve.

General Procedure in the Solution of Problems.

The following instructions should be noted.

(i) Read the question very carefully. Don't start writing till you are sure you understand what it is that you are given and what you are asked to find out.

(ii) Take a letter to stand for some unknown number which the problem involves and state carefully exactly what this letter represents.

E.g. *Never say*: Let the man's age be x.

A clear statement would be: Let the man be x years old now.

Never say: Let the price of the tea be x.

A clear statement would be: Let 1 lb. of tea cost x shillings.

Remember that a letter represents a number, it does not represent a number-of-things.

(iii) Check your answer by using the actual data of the problem. It is not sufficient to check by substituting in the equation, because your equation may be wrong.

Example XII. I think of a number, multiply it by 4 and then add 9. The result is 37. What is the number?

Suppose I think of a number x.

If I multiply x by 4 and then add 9, the result is $4x+9$.

$$\therefore\ 4x+9=37;$$

$$\therefore\ 4x=37-9=28;$$

$$\therefore\ x=7;$$

$\therefore$ the number thought of was 7.

Check: I think of 7; multiplying by 4, I obtain 28; adding 9, I obtain 37.

EXERCISE IV. *f*.

Solve the following problems by algebra, and check each answer.

1. I think of a number, divide it by 4 and add 11; the result is 17. What is the number?

2. I think of a number, add to it one-third of itself; the result is 28. What is the number?

3. The result of adding 42 to a certain number is the same as multiplying that number by 4. What is the number?

4. If I halve a certain number and add 1, the result is the same as dividing by 3 and adding 4. What is the number?

5. The sum of two consecutive numbers is 55. What is their product?

6. What number exceeds 17 by the same amount as it falls short of 55?

7. The sum of three consecutive even numbers is 72; what are they?

8. From three-quarters of a certain number, 3 is subtracted; the result is two-thirds of that number. What is the number?

9. I think of a number, add 2 to it, multiply the sum by 5 and then subtract 7; the result is 23. What is the number?

10. I think of a number, double it, add 12 to the result and then divide by 2. I next subtract the number I first thought of; the result is 6. Can you find the original number?

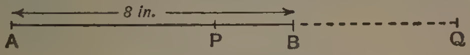


FIG. 100.

11. In Fig. 100, find AP if AP is 3 inches longer than PB.

12. In Fig. 100, find PB if AP is twice the length of PB.

13. In Fig. 100, find AQ if AQ is $2\frac{1}{2}$ times as long as BQ.

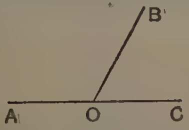


FIG. 101.

14. In Fig. 101, $\angle AOB = 2\angle BOC$, find $\angle BOC$ in degrees.

15. In Fig. 101, $\angle AOB$ exceeds $\angle BOC$ by $\frac{4}{5}$ of a right angle, find $\angle BOC$ in degrees.

16. The flagstaff PQ in Fig. 102 is one-fifth of the height AB of a tower ; Q is 90 ft. above the ground. What is AB ?

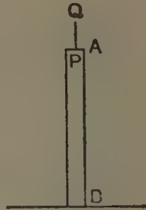


FIG. 102.

17. In Fig. 103, find $\angle B$ if $\angle B = \angle C = 2\angle A$.

18. In Fig. 103, find $\angle A$ if $\angle A = \frac{2}{3}\angle B = \frac{4}{5}\angle C$.

19. In Fig. 103, find $\angle A$ if $\angle B - \angle C = 10°$ and $\angle A - \angle B = 25°$.

20. In Fig. 104, find $\angle P$ if $\angle P = 3\angle Q$.

21. Where must a pole 40 ft. long be cut, so that one part is two-thirds of the length of the other ?

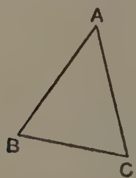


FIG. 103.

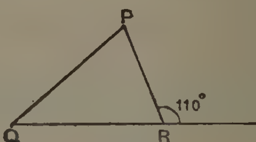


FIG. 104.

22. Divide 10 shillings between two boys, so that one receives a shilling more than the other.

23. In the $\triangle ABC$, $\angle B = 90°$ and $\angle C$ is two-thirds of $\angle A$, find $\angle C$.

24. The total number of children in a school is 266 ; there are 38 more girls than boys ; find the number of girls.

25. It costs eight times as much to buy a house as to furnish it. The total cost for both is £3600. What is the cost of furnishing ?

26. If the day is 3 hours longer than the night, find the length of each.

27. The sum of the angles of an n-sided figure is $(2n-4)$ right angles. How many sides has a figure if the sum of its angles is 10 right angles ?

28. Fig. 105 represents a hurdle whose width is $1\frac{1}{2}$ times its height; it is made of metal strips and the total length of metal required is 36 ft. How wide is the hurdle?

FIG. 105.

29. Fig. 106 represents a skeleton wire cage; AB is twice as long as BC and as CD; the total length of wire is $5\frac{1}{2}$ ft. What is the length of AB?

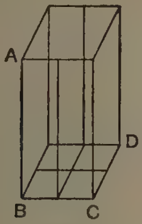


FIG. 106.

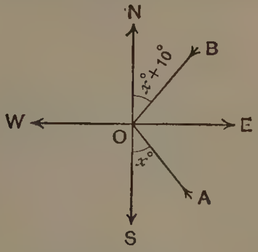


FIG. 107.

30. A man, aged 40, has a son, aged 9. When will the man be just twice as old as his son?

31. The wind backs from direction AO to direction BO (see Fig. 107), a change of direction of 100°. Find its first direction.

32. A lead sheet, 6 ft. wide, of indefinite length, is bent to form a gutter; Fig. 108 represents its cross-section; the base BC is $2\frac{1}{2}$ times each of the sides AB, CD. Find the area of the cross-section in sq. inches.

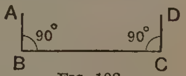


FIG. 108.

33. Find x if $3x$ minutes past five is the same time as $2x$ minutes to six.

34. One tank contains 24 gallons of water and another contains 5 gallons; equal quantities of water are pumped into each tank. How much has been added when one tank contains just twice as much as the other?

FIG. 109.

35. In Fig. 109, O is the mid-point of AB and AP is one-fifth of PB; OP is 2 inches, find AB.

36. In Fig. 110, $\angle A$ is half each of the other angles. Find $\angle A$. [Use the fact stated in No. 27.]

37. Use the statement in No. 27, to find the number of sides of a figure, if each of its angles is 156°.

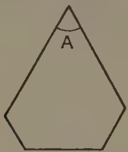


FIG. 110.

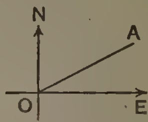


FIG. 111.

38. Of the 2000 people in a town, 200 are soldiers. How many extra soldiers must be brought in to make the number of soldiers one-fifth of all the people in the town ?

39. In Fig. 111, the direction OA is $x°$ E. of N.; it is also $(x-24)$ degrees North of East. What is its direction ?

40. If, with summer time, the Sun rises at x a.m. and sets at $(x+2)$ p.m., and if mid-day is 1 p.m., find x.

41. For an excursion, the railway fare is one-quarter of the ordinary fare. You save 5s. 6d. by taking an excursion ticket. What is the ordinary fare ?

42. Two milk cans together contain 50 pints of milk. When 2 pints are taken out of one, there remains in it half of what the other contains. How much was in the other ?

43. If a boy's weekly wages are raised half-a-crown, he receives for five weeks one shilling less than he used to get for six weeks. What are his new wages ?

44. If a shopkeeper sells a hat for 14s. his profit is three times as much as his loss would be if he let it go in a sale for 8s. What did the hat cost the shopkeeper ?

45. A grocer buys some eggs at 2d. each ; he finds that one dozen are bad ; he sells the rest at 3d. each and makes 5s. profit. How many did he buy ?

46. By selling oranges at 2s. 4d. a dozen, a grocer makes a profit of $\frac{3}{5}$ of the cost price. What did the oranges cost per dozen ?

47. I bought 20 pencils for 3s. ; some cost $1\frac{1}{2}$d. each and the rest $2\frac{1}{2}$d. each ; how many of the cheaper kind did I buy ?

48. How can you share 12 shillings between two boys, so that one gets half-a-crown more than twice the amount the other gets ?

49. 5 lb. of tea at a certain price is mixed with 4 lb. of tea costing 9d. per lb. more. The average price of the mixture is 2s. 7d. per lb. Find the price of each kind.

50. A man buys one lot of eggs at 1s. 6d. a dozen and a second lot, which is 3 dozen more than the first lot, at 2s. a dozen; he sells them all at 2s. 6d. a dozen and makes 15 shillings profit. How many eggs did he buy altogether ?

Harder Problems.

It often happens that a problem appears difficult because the pupil does not fully *understand and keep in mind* exactly what is given. On this account, the two following suggestions are worth making :

(i) Whenever possible, make a rough diagram and show the data on it.

(ii) Having chosen a letter for the unknown and stated precisely what this letter represents, re-write the question, using this letter to make the statement of the problem more detailed (see the Examples below). It may be necessary to do this several times.

Example XIII. A man buys 15 copies of a book for £3 3s., some of them bound in leather at 5s. each and the rest in cloth at 3s. each. How many leather-bound copies did he buy ?

Suppose he buys x copies bound in leather.

[Now re-write the question, using more detail.]

A man buys 15 copies of a book for 63 shillings; he buys x copies at 5s. each and $(15-x)$ copies at 3s. each.

[Or, writing it out again]:

A man buys 15 copies of a book for 63 shillings; he buys x copies for $5x$ shillings and $(15-x)$ copies for $3(15-x)$ shillings.

$$\therefore\ 5x \text{ shillings} + 3(15-x) \text{ shillings} = 63 \text{ shillings};$$

$$\therefore\ 5x+3(15-x)=63;$$

$$\therefore\ 5x+45-3x=63;$$

$$\therefore\ 2x=63-45=18;$$

$$\therefore\ x=9.$$

$\therefore$ **he buys 9 copies bound in leather.**

Check : Apply the answer to the actual problem.

9 copies at 5s. each cost 45s.

15 − 9 = 6 copies at 3s. each cost 18s.

$\therefore$ total cost = 45s. + 18s. = 63s. = £3 3s.

Example XIV. A man leaves his house at 9 a.m. and walks to a town at 4 miles an hour. He spends two hours there and then walks back at 3 miles an hour, arriving home at 6 p.m. How far is the town from his house?

Let the town be x miles from his house.

[Now show the data on a figure.]

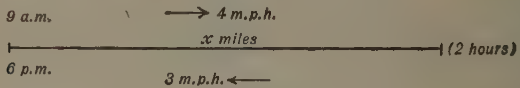


FIG. 112.

[Now re-write the question in detail.]

A man leaves his house at 9 a.m. and walks x miles to a town at 4 m.p.h., this takes $\frac{x}{4}$ hours; he waits 2 hours; he then walks back x miles at 3 m.p.h., this takes $\frac{x}{3}$ hours; he arrives home at 6 p.m., which is 9 hours after he started.

$$\therefore \frac{x}{4} \text{ hours} + 2 \text{ hours} + \frac{x}{3} \text{ hours} = 9 \text{ hours};$$

$$\therefore \frac{x}{4} + 2 + \frac{x}{3} = 9;$$

$$\therefore 3x + 24 + 4x = 108;$$

$$\therefore 7x = 84; \quad \therefore \quad x = 12;$$

$\therefore$ the town is 12 miles from his house.

Check: Apply the answer to the actual problem.

To walk 12 miles at 4 m.p.h. takes 3 hours; he waits 2 hours; to walk back 12 miles at 3 m.p.h. takes 4 hours.

$$\therefore \text{ total time} = (3 + 2 + 4) \text{ hours} = 9 \text{ hours}.$$

From 9 a.m. to 6 p.m. is 9 hours.

EXERCISE IV. *g*.

1. Fig. 113 represents a rectangular enclosure $ABCD$; $DE = 2EC$ and $AD = 2AB$; it is 40 yards longer from A to E one way round than the other. Find the length of each side.

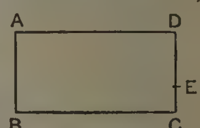


FIG. 113.

2. One man starts with a salary of £150 a year, which is increased by £20 a year at the end of each year. At the same time another man starts at £225 a year and has an annual increase of £7 10s. In what year will their salaries be equal?

3. There are 220 people at a concert; some pay 1s. and the rest 2s. each. The takings amount to £13. How many bought 1s. tickets?

4. In a mill, the men get 7s. 6d. a day and the women get 6s. a day; the number at work is 200 and the wages amount to £69 a day. How many men are employed?

5. Fig. 114 represents a cyclist riding from A to B and back again; the double journey takes 5 hours; find x.

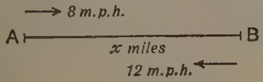


FIG. 114.

6. A table is marked at £P, but there is a discount of 3d. in the shilling for cash; the cash price is £12. What is P?

7. A man rows up-stream at 3 miles an hour and back to the same place at 5 miles an hour, and takes 48 minutes altogether. How far up-stream did he go?

8. Fig. 115 represents a cyclist leaving A at the same moment as a pedestrian leaves B. How far from A will they meet?

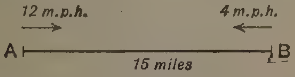


FIG. 115.

9. A kettle of water is placed on a stove; after t minutes its temperature is $(16+5t)$ degrees Centigrade. How long will it take to boil?

What was its temperature at the beginning?

10. A man buys 800 bulbs for two guineas, some of them at 25 for a shilling and the rest at 6s. a hundred. How many were there of the more expensive kind?

11. The perimeter of a rectangle is 5 ft.; another rectangle, twice as long and half as broad, is 2 ft. more in perimeter. What are their areas?

12. At a fair, a boy receives 8d. for a hit and pays 3d. for a miss; he has 24 shots and has to pay 6d.; how many hits did he score?

13. The lengths of three sides of a rectangle are shown in Fig. 116, the units being inches. Find (i) the fourth side, (ii) the perimeter, (iii) the area.

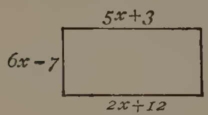


FIG. 116.

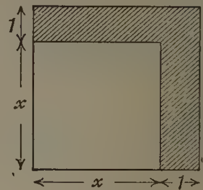


FIG. 117.

14. Use Fig. 117 to find two consecutive whole numbers whose squares differ by 37.

15. A river flows at 3 miles an hour. What is the speed through the water of a boat that can go down-stream twice as fast as up-stream?

16. There are steps, each 8 in. high, from A to B, in Fig. 118; some are 12 in. wide and the rest 15 in. wide. How many of them are 12 in. wide?

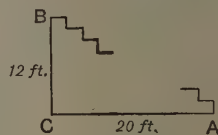


FIG. 118.

17. If I walk to the station at 4 m.p.h. to catch a train, I have 3 minutes to spare; but if I walk at 3 m.p.h., I shall miss the train by 1 minute. How far off is the station?

18. If $C°$ Centigrade represents the same temperature as $F°$ Fahrenheit, $C = \frac{5}{9}(F - 32)$, what temperature on the Fahrenheit scale corresponds to (i) 0° C., (ii) 100° C.? What is the temperature which is recorded by a number twice as large on the Fahrenheit scale as on the Centigrade scale?

19. At a concert, 100 people paid half-a-crown each and the rest paid 6d. each. The average paid per head was a shilling. How many tickets were sold?

20. A publisher offers to pay an author either 7d. on every copy of his book that is sold after the first 500 copies, or 5d. on every copy sold. The author accepts the first offer and gains by doing so. What can you say about the number of copies sold?

Subject of a Formula.

The reader is by now familiar with the use of Formulae and has had some experience in constructing them. The Example on p. 87 gives the formula for calculating the simple interest £I on a sum of money £P lent for T years at R per cent. per annum :

$$I = \frac{PRT}{100}.$$

We call I the **subject** of this formula.

Suppose, however, the problem was reversed, as follows :

Example XV. If the simple interest on a sum of money, £P, lent for T years is £I, calculate the rate per cent. per annum, R.

In this case, our formula should express R in terms of P, T, I.

We have $\qquad \frac{PRT}{100} = I.$

Multiply each side by 100 ; $\quad \therefore\ PRT = 100I.$

Divide each side by PT ; $\quad \therefore\ \mathrm{R} = \frac{100\mathrm{I}}{\mathrm{PT}}.$

We have now obtained a formula with R as its *subject* : and the process is called **changing the subject of the formula.** We were given a formula with I as subject, and have altered it so as to make R the subject, *by using precisely the same methods as are employed in solving equations.*

Exercise IV. *h*.

1. What relation connects x and y in Fig. 119 ? Make (i) y, (ii) x the subject of the formula.

2. The perimeter of Fig. 119 is p in. ; what formula connects a, b, p ? Make b the subject.

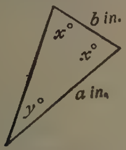


FIG. 119.

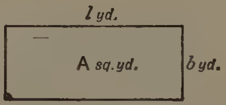


FIG. 120.

3. Fig. 120 represents a rectangular field of perimeter p yd. Find formulae for the following cases :

(i) Given l, b, make A the subject.
(ii) Given A, l, make b the subject.
(iii) Given l, b, make p the subject.

(iv) Given l, p, make A the subject.

(v) Given b, A, make p the subject.

4. Fig. 121 represents the four walls of a room folded out flat, the units are feet. The total area of the walls is A sq. ft. Express A in terms of l, b, h and then make (i) h, (ii) b, the subject of the formula.

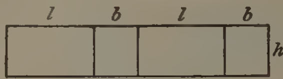


FIG. 121.

5. Interpret geometrically the formula $A=\frac{1}{2}b \,.\, h$ and then make h the subject. What is h if $A=3\frac{1}{2}$, $b=5$?

6. Interpret geometrically the formula, (i) $V=l\,.\,b\,.\,h$; (ii) $V'=\dfrac{l\,.\,b\,.\,h}{27}$. Make h the subject in each case.

7. If $F°$ Fahrenheit is the same temperature as $C°$ Centigrade then $F=32+\dfrac{9C}{5}$. Make C the subject of the formula and find the value of C, when (i) $F=32$, (ii) $F=212$.

8. Interpret the formula $W=\dfrac{5bd^2}{4l}$ (see Ex. I. *h*. No. 24); then make (i) b, (ii) l, (iii) d the subject of the formula.

9. Interpret the formula $A=P+P\,.\,\dfrac{R}{100}$ and make R the subject.

10. Fig. 122 shows a symmetrical cross; its area is A sq. in. Express A in terms of b, c and make c the subject. What is c, if $A=108$, $b=3$?

11. The edge of a cube is l in., the total area of its surface is A sq. in.; express A in terms of l and then make l the subject. What is l if $A=96$?

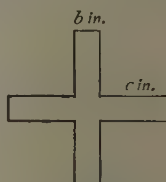


FIG. 122.

12. The area of a trapezium is given by the formula

$$A=\tfrac{1}{2}h(x+y).$$

Draw freehand a figure to show what the letters represent. Make (i) h, (ii) x the subject.

13. (i) Interpret the formula $C = 2\pi r$ and make r the subject.

(ii) Fig. 123 represents a semicircular disc of perimeter p in.; express p in terms of π, d and then make d the subject; take $\pi = \frac{22}{7}$.

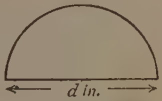


FIG. 123.

14. If a train of weight W tons is travelling round a curve of radius r ft. at v ft. per sec., the rails must exert on the train a horizontal force of P tons towards the centre of the curve, given by $P = \frac{Wv^2}{32r}$. Make v the subject.

15. Power is being transmitted by an endless belt running round a shaft; if the tensions in the belt on the two sides of the shaft are T_1 lb. and T_2 lb., and if the belt is travelling at v ft. per sec., the horse-power H transmitted is given by the formula $H = \frac{v(T_1 - T_2)}{550}$. Make T_1 the subject.

16. Interpret the formula $A = \pi r^2$ and make r the subject.

17. Interpret the formula $V = \pi r^2 h$ and make (i) h, (ii) r the subject.

18. The area of the total surface of a closed circular cylinder of height h in. and base-radius r in. is A sq. in.; show that $A = 2\pi r(r + h)$ and make h the subject.

19. (i) What is the number whose square root is 7?

(ii) What is N if $\sqrt{N} = 8$? (iii) What is A if $\sqrt{A} = l$?

(iv) What is A if $r = \sqrt{\frac{A}{\pi}}$?

20. From a masthead h feet above the surface of the sea it is possible to see R miles, where $R = \sqrt{\left(\frac{3h}{2}\right)}$. Make h the subject of this formula. What is h, if $R = 12$?

21. Interpret the formula $H = \frac{9Av^3}{10^7}$ (see Ex. S. 2, p. 37, No. 6); then make (i) A, (ii) v the subject.

22. Interpret the formula $v = 8\sqrt{H}$ (see Ex. S. 2, p. 36, No. 3); then make H the subject. What is H if $v = 20$?

SUPPLEMENTARY EXERCISE. S. 5

1. A first-class ticket from Winchester to Dorchester is two-thirds as much again as a third-class ticket. The cost of 8 first-class and 20 third-class tickets is £15. What is the cost of each kind ?

2. On a certain railway journey, 60 lb. of luggage is free and the charge for every lb. above that is $\frac{1}{2}$d. How much luggage do I take if I pay at the rate of 1d. per 5 lb. on the whole of it ?

3. The sides of a triangle are proportional to 3, 4, 5 ; the perimeter is 18 inches. What is the length of the shortest side ?

4. A gardener reckons that, for every gallon of potatoes he plants, he will get a crop of 7 gallons. How much must he plant so as to have 150 gallons to eat and enough left over to plant the same amount next year ?

5. In Rugby football, a try counts 3 points and a goal 5 points. The number of goals scored by Red was one less than the number of tries scored by Red. White scored twice as many tries as Red, but only half as many goals. The final scores differed by 4 points. Who won and with what score ?

6. A flag (see Fig 124) is made by arranging a coloured cross symmetrically on a white rectangular sheet $ABCD$; the width of each arm of the cross is 3 ft., $AB = 12$ ft., and the area of the cross is half the area of $ABCD$. Find BC.

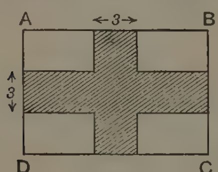


FIG. 124.

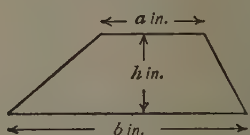


FIG. 125.

7. How much tea at 2s. 6d. per lb. must be mixed with 12 lb. of tea at 1s. 10d. per lb., so that the mixture may be worth 2s. 3d. per lb. ?

8. Find v if v miles per hour is the same speed as $(v + 3\frac{1}{2})$ feet per second.

9. The area of the trapezium in Fig. 125 is $\frac{1}{2}h(a+b)$ sq. in. Express this in terms of h, if $6a=2b=3h$.

10. What must be a man's income if he has £450 a year after paying tax at the rate of 4s. in the £ on the part of it above £150 ?

11. What is the angle between the hands of a clock at 4 o'clock ? What extra angle has the hour hand turned through when the minute hand has turned through $x°$ and what is the angle between the hands at this moment ? What is x, if the hands are now at right angles ? What is the corresponding time ? At what time are the hands at right angles between 4.30 and 5 o'clock ?

FIG. 126.

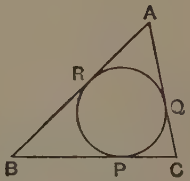


FIG. 127.

12. Since the tangents from a point to a circle are equal, it follows that, in Fig. 127, $AQ=AR$, $BR=BP$, $CQ=CP$. If $AC=4$ in., $AB=5$ in., $BC=7$ in., $AQ=x$ in., find x.

13. What is the number of two digits with x for the ten digit and 5 for the unit digit ? If the number equals five times the sum of its digits, find what the number is.

14. Two tanks (see Fig. 128) have horizontal bases 4 ft. square and 3 ft. square ; they are connected by a pipe con-

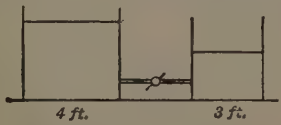


FIG. 128.

trolled by a tap. When the tap is closed the level in the larger tank is $2\frac{1}{2}$ ft. above that in the smaller tank. How much does the level in the larger tank fall when the tap is opened ?

15. The same number is added to the numerator and denominator of the fraction $\frac{5}{12}$ and the result equals $\frac{2}{3}$. Find the number.

16. A string l in. long is attached to the ends B, C of a rod BC, p in. long, which is loaded at C, so that when the string is slung over a small peg A, the rod BC hangs vertically, as shown in Fig. 129; the straight portions of the string may also be regarded as vertical. Find the depth below A of (i) C, (ii) B, (iii) the mid-point of BC.

FIG. 129.

17. A man pays 4s. in the £ income-tax on the part of his income above £160. Next year, with the same income, he pays 4s. 6d. in the £ on the part above £120. His tax is £20 more the second year than the first. What is his income?

18. A man has £a in one bank and £b in another. He has £c to deposit. How must he divide the money to make the two accounts equal? Is it always possible?

19. A slow train passes a level crossing travelling 24 miles per hour; ten minutes later an express travelling 60 miles an hour passes the crossing in the same direction. How far from the crossing will the first train be overtaken?

20. From Croydon to Brighton is 40 miles. A starts from Croydon to walk to Brighton at $3\frac{1}{2}$ miles an hour; B starts half an hour later from Brighton to cycle to Croydon at 10 miles an hour. How far from Croydon do they meet?

21. A certain sum of money is sufficient to pay A's wages for 20 days or B's wages for 30 days. For how long will it suffice to pay A and B, if both are at work?

22. A lump of ore weighing 15 lb. contains 10 per cent. of silver and the rest is lead. How much lead must be removed so that the remainder may contain 80 per cent. of silver?

23. The nth number in the set of numbers 3, 7, 11, 15, 19, ... is $4n-1$. Show that this is true for the first four numbers. Does 299 belong to the set? If so, where does it come?

24. A man has £100 and invests it at 4 per cent. simple interest; another man invests £80 at 6 per cent. simple interest. If neither spends the interest, when will they have equal amounts?

25. In Fig. 130, express x in terms of a, b, c.

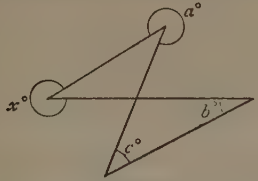


FIG. 130.

26. Two men A, B have the same income. A saves one-fifth of his income; B spends £300 a year more than A spends. In 5 years, B spends £300 more than he receives. What is the income of each?

MISCELLANEOUS EXAMPLES

M. II

1. What is the angle (in degrees) between the hands of a clock at (i) 1 o'clock, (ii) 5 o'clock, (iii) x o'clock, if x is an integer not greater than 6, (iv) x o'clock, if x is an integer between 6 and 12?

2. A ton of coal costs x shillings in July, but y shillings in December. How much does a man save (in £) if he buys 60 tons in July instead of December?

3. A tank contains V cu. ft. of water. It is 6 ft. long and 3 ft. wide, what is the depth of the water in inches?

4. For what value of x is $3x - 2$ equal to $2x + 5$?

5. The average age of n boys is 15: the eldest is 18 and the youngest is 10; what is the average age of the rest?

6. For what value of x are $\frac{x}{10}$ and $\frac{x}{8}$ consecutive whole numbers?

7. A man's annual salary increases by the same amount every year. The first year, it is £a; the second year, it is £b; how much is it (i) the third year, (ii) the tenth year, (iii) the nth year?

8. If 25 (Swiss) francs are worth £1, how many shillings shall I get for f (Swiss) francs?

9. How many florins have the same value as y half-crowns?

10. The perimeter of a square is $2a$ inches; what is its area?

11. The pressure in a boiler is p grams per sq. cm.; express this in kg. per sq. metre.

12. Find x in Fig. 131.

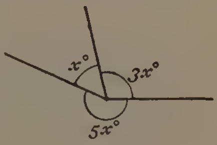


FIG. 131.

13. What is x, if $\frac{1}{x}+\frac{1}{2x}=\frac{1}{3}$?

14. A penny weighs $\frac{1}{3}$ oz., a halfpenny weighs $\frac{1}{5}$ oz. Show that n pennies and n halfpennies together weigh more than $\frac{n}{2}$ oz.

15. When eggs are P pence per dozen, how many do you get for a shilling ?

16. The total perimeter of a semicircular area is 6 ft. What is the radius ? [Take $\pi=\frac{22}{7}$.]

17. A roll of wall-paper is 12 yd. long and 21 in. wide. How many rolls are required to cover $35p$ sq. yd. of wall ?

18. How many days are there *between* the pth day of July and the qth day of August ?

19. A verst $=\frac{2}{3}$ mile ; how many versts are there in n miles ?

20. A map is l in. long and w in. wide ; its scale is 6 in. to the mile. How many sq. miles does it represent ?

21. A sovereign is composed of $\frac{11}{12}$ gold, $\frac{1}{12}$ alloy, as regards weight. If the gold in a sovereign weighs w grains, what is the weight of the sovereign ?

22. Divide £1 between two boys so that one receives half-a-crown more than the other.

23. When it is 12 o'clock at Petrograd it is 10 o'clock at Greenwich. What is the time at Greenwich when it is x o'clock at Petrograd, (i) if $x>2$, (ii) if x is 2 or less than 2 ?

24. A's income is $\frac{1}{2}$ of B's and is $\frac{2}{3}$ of C's ; B's income is £225 more than C's. What is A's income ?

25. A can just give B a start of 2000 points in 9000 at billiards. When B's score is x, what ought A's score to be ? When A's score is y, what ought B's score to be ?

26. The average of four numbers is p, and the average of three of them is q. What is the fourth number ?

27. In Fig. 132, P is three times as far from A as it is from B, and $QB=\frac{2}{3}QA$. If $PQ=9$ in., what is AB ?

FIG. 132.

28. x is the smallest prime number greater than 50, and y is the largest prime number less than 50, prove that $8x-9y=1$.

29. How many seconds will a train travelling at x feet a second take to pass completely through a station l yd. long, the length of the train being t yd. ?

30. A florin weighs $\frac{2}{5}$ oz.; how many florins together weigh W oz.?

31. If a man sold his bicycle for £a he would lose 10 per cent. on the cost price; if at £b he would gain 10 per cent. on the cost price. Find a in terms of b.

32. The section of a tunnel is of the shape of a semicircle above a rectangle (see Fig. 133). If $h = \frac{3}{4}d$. and $d = 20$, find the perimeter of the section.

Find also an expression for the perimeter in terms of d, if $h = \frac{3}{4}d$.

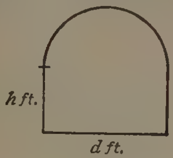


FIG. 133.

33. A man reckons that the cost of growing potatoes is 2s. per bushel. He keeps 12 bushels for his own use and pays his expenses by selling the rest at 2s. 4d. per bushel. How many bushels did he grow?

34. If $y = 11x + 71$, find the increase in the value of y when x increases from 17 to 27.

35. Find z, if $\frac{9}{z} = \frac{63}{17}$.

36. A river flows at 3 miles per hour. At what rate does a man row up-stream, if with the same effort he travels four times as fast coming down-stream.

37. If $x = 3y$ and $y + 2z = 8$, what is the value of xy when $z = 3$?

38. If $\frac{2x-1}{5}$ and $\frac{x-1}{2}$ represent consecutive integers, what integers are they?

39. A coal merchant has contracted to deliver 200 tons of coal at 28s. a ton, but, when the time comes, finds that his stock, which has cost 21s. a ton, is insufficient. He has therefore to buy more coal and this costs 32s. a ton, and so he loses £18 on the transaction. How much coal did he have to buy?

40. State in detail what the following instruction means: Work out in this exercise each question whose number is of the form $5n - 3$, where n is an integer.

How many questions does this involve, in this exercise?

CHAPTER V

GRAPHS OF STATISTICS

THE following example is intended for oral work.

Example I. A motor-car is fitted with a gauge which shows the number of gallons of petrol in the tank. When full, the tank

Petrol gauge readings.

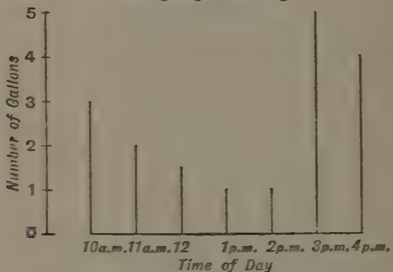


FIG. 134.

holds 5 gallons. A motorist starts out at 10 a.m. and notes the readings on the gauge at hourly intervals. The result is shown in Fig. 134.

EXERCISE V. *a*. (Oral.)

Use the data and figure of Example I., above, to answer the following questions :

1. What is the length in inches of the line which represents (i) 2 gallons, (ii) 5 gallons ?

2. How much petrol did he start with ? How much had he at 11 a.m. and at noon ?

3. During what time is it probable that the car was not running ?

4. About what time did he fill up the tank ?

5. If the car averages 24 miles to the gallon, estimate the distances travelled in successive hours.

6. (i) Would it be possible to attach a meaning to an upright line inserted midway between the second and third uprights ?

(ii) If so, could you without any further data, insert any additional uprights in the figure, which would represent the facts with fair accuracy ?

It will be noted that Fig. 134 is drawn without the use of squared paper, and it is suggested that the examples in Exercise V. *b* should be taken in the same way. The *initial use of plain paper* for graphical work serves two purposes: (i) it emphasises the fact that the only object of using squared paper is to save time, both when plotting and reading off values; (ii) it impresses on the pupil the significance of the scale to be selected and forces him to interpret it when making his own measurements with a graduated ruler.

At first, all measurements up the page should start from zero. But when Exercise V. *b*, No. 4, has been drawn in this way, the pupil will see for himself, or may be helped to see, that space can be saved, or a larger scale employed, by starting higher up.

The diagrams of this chapter are drawn in a column-form. In each case the pupil should consider, (i) whether any meaning can be attached to interpolated upright lines, (ii) whether, in such a case, interpolation is possible with fair accuracy without further data. In this way, the ground is prepared for the idea of functionality and the graphs of continuous functions. It is suggested that throughout this chapter the tops of the uprights should *not* be joined either by straight lines or curves, so that attention may be concentrated on the uprights themselves.

Exercise V. *b*.

Draw, on *plain* paper, diagrams as in Fig. 134 to represent the records tabulated below. State in each case, (i) whether any meaning can be attached to intermediate upright lines, (ii) whether interpolation is possible with fair accuracy without further data.

Give each diagram a title and write along each axis what that axis represents and, where suitable, how it is graduated.

1. The following table shows the distribution over the world of the chief languages :

Language.	English.	German.	Russian.	French.	Spanish.
Number who speak it.	160,000,000	100,000,000	100,000,000	70,000,000	50,000,000

2. The following table shows the average diameter of oak trees of different ages :

Age in years	10	20	30	50	70	100	150	200
Diameter in inches	5	10	14	23	32	41	54	64

3. A shopkeeper finds that his average daily receipts are as follows :

Day	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Receipts in £	13	11	14	6	12	18

4. The following table shows the average weight of boys of different ages :

Age in years	11	12	13	14	15	16	17	18
Weight in lb.	79	85	92	102	114	129	142	146

Example II. Full marks each week in a class is 100; the following table shows the marks scored by a boy in successive weeks of a term:

Number of week	1	2	3	4	5	6	7	8	9	10	11
Number of marks	56	62	81	54	60	52	78	83	65	70	72

Represent this table by a graph.

We draw a line across the squared paper, on which we mark points at equal intervals to represent the 1st week, 2nd week, 3rd week, etc. We then draw a line up the page, which we graduate to represent the marks scored ; since no mark is less than 50 or greater than 90, it is unnecessary to show any graduation outside these limits. These two lines are called the **axes of reference** and the graduations show the **scales** which have been selected. We then draw lines up the page whose lengths represent the marks scored in the successive weeks.

Oral work, using Fig. 135.

(i) In which weeks did he obtain more than 75 marks ?

(ii) In which weeks did he obtain less than 60 marks ?

(iii) In which week did he obtain (*a*) most marks, (*b*) fewest marks?

(iv) If an upright line was drawn midway between the 3rd and 4th uprights, would it have any meaning ?

The following instructions are important:

(i) Write above the diagram or graph a *short* statement of what the graph records. Every graph should have a title or heading.

(ii) Write along each axis what that axis represents.

Weekly Marks.

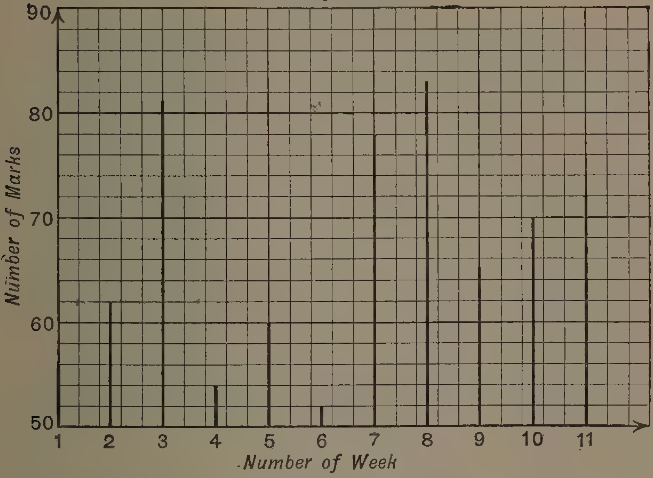


FIG. 135.

(iii) Graduate each axis so as to show clearly the scale selected; but before doing so, consider carefully what range of values is to be represented. Thus, in the above example, the marks obtained are never less than 50 or more than 90; the lowest graduation on the upright axis is therefore taken as 50 (it would be a waste of space and involve a needlessly small scale to start from 0), and the highest is taken as 90, not as 100.

(iv) Choose as large a scale as the paper will allow, but the scale should be one which makes both plotting and reading easy.

(v) A graph records how one quantity varies in size when another quantity varies. The values of one quantity are selected and then the corresponding values of the other quantity are observed or calculated.

The axis for the quantity whose values are selected should always be drawn across the page. The axis for the quantity whose values are observed or calculated should be drawn up the page. Thus, in the example on p. 122, we *select* special times and then *observe* the height in the gauge at these times. The axis showing the times

is therefore drawn across the page, the axis for the readings made at these times is drawn up the page. In other words, the axis across the page represents the *independent variable* and the axis up the page represents the *dependent variable*.

Exercise V. *c*.

Represent on squared paper the following statistics, as in Fig. 135. State in each case (i) whether any meaning can be attached to intermediate upright lines, (ii) whether interpolation is possible with fair accuracy without further data.

1. The number of motor-cars sold by a firm in successive quarters is as follows :

	1925				1926			
Period - -	I	II	III	IV	I	II	III	IV
Number of cars	416	564	623	510	437	624	735	576

2. The number of pupils who left and entered a school during successive years was as follows :

Year - - -	1920	1921	1922	1923	1924	1925	1926
Number leaving -	164	147	173	169	152	144	177
Number entering -	153	165	171	156	174	168	180

3. Using the data of No. 2, if there were 810 pupils in the school at the end of 1919, show graphically the total numbers of the school at the end of the years 1920-1926.

4. The population of Australia is recorded as follows :

Year - - -	1871	1881	1891	1901	1911	1921
Population in millions	1·66	2·25	3·17	3·77	4·45	5·44

Estimate the population in 1896 and 1907.

5. The annual premium for a Life Assurance of £1000 varies with the age of the insurer at the time of the first payment, as follows :

Age -	25	30	35	40	45
Premium -	£14 3s.	£16 8s.	£19 7s.	£23 1s.	£27 17s.

Estimate the premiums for ages of 32 and 38.

6. The wheat production of the United Kingdom is shown in the following table :

Year - - -	1912	1914	1915	1916	1917	1918	1919	1920	1925
Quarters of wheat in millions - -	7·18	9·80	7·24	7·47	8·04	11·64	8·66	7·10	7·00

7. The following table shows the length of the longest day in different latitudes :

Latitude in degrees -	15	25	35	45	50	55	60	65
Length of day in hours	12·9	13·6	14·4	15·4	16·1	17·1	18·5	21·2

Estimate the length of the longest day in latitudes 30°, 52°.

8. The mean temperatures in degrees Fahrenheit for the various months are as follows, for London :

Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
39·6	40·7	43·1	48·2	54·1	59·7	63·2	62·5	58·2	51·2	45·0	41·3

9. In Sept. 1926, Civil Service annual salaries were supplemented by a bonus on the following scale :

Salary in £ - -	90	200	300	400	600	800	1000
Bonus in £ - -	63	99	123	147	176	195	205

10. The following table shows the length of the day on the 1st of each month in 1926, at Greenwich :

Jan.	Feb.	Mar.	Apr.	May.	June.
7 h. 51 m.	9 h. 4 m.	10 h. 48 m.	12 h. 50 m.	14 h. 44 m.	16 h. 13 m.

July.	Aug.	Sept.	Oct.	Nov.	Dec.
16 h. 30 m.	15 h. 24 m.	13 h. 35 m.	11 h. 38 m.	9 h. 40 m.	8 h. 9 m.

Estimate the length of day on May 15, October 16.

11. The income-tax a man paid in successive years was as follows :

Year - -	1919	1920	1921	1922	1923	1924	1925	1926
Tax in £ -	212	225	247	203	235	192	208	195

12. The following table compares the death-rate of first-class cricketers with that of other men by giving the number per 1000 who die between various ages:

Age - -	25-35	30-40	35-45	40-50	45-55	50-60
Cricketers -	35	46	61	83	117	122
Others - -	47	60	78	102	138	190

PART II

EASY REVISION PAPERS. A. 6-15

A. 6

1. Criticise and correct the following statement: "If the price of 1 lb. of coffee is x, the price of 2 lb. is $2x$."

2. Fill in the blanks in the following:

$$\text{(i) } \frac{a}{x}=\frac{}{2x}; \quad \text{(ii) } \frac{a}{c}=\frac{a^2}{}; \quad \text{(iii) } x=\frac{}{2}=\frac{}{y}.$$

3. Simplify (i) $(x+a)+x+(x-a)$;
(ii) $4(x^2-2x+1)+2(x^2+2x+1)$.

4. A class of boys is told to answer questions 2, 5, 8, 11, 14, etc. in an exercise. Find a formula for these numbers.

5. In Fig. 136, find x.

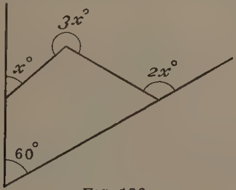


FIG. 136.

A. 7

1. If a rectangle is $(x+2)$ ft. long and $(x+1)$ ft. wide, the area is (x^2+3x+2) sq. ft. Show that this is true when $x=6$.

2. Find a formula for the numbers 13, 23, 33, 43, 53, 63, etc.

3. Subtract (i) $x-(y+2)$ from $x+(y-2)$; (ii) $(b^2-2b)\div b$ from $(a^2-2a)\div a$.

4. For what value of t is $\frac{2}{t-1}$ equal to $\frac{5}{t}$, and what is the value of each in this case?

5. x metres equals approximately x yd. $+\frac{x}{12}$ yd. $+\frac{x}{100}$ yd. Express 1200 metres in yd. and express z yd. in metres.

A. 8

1. Prove that $a^6 \times a^2 = a^8$. Write down, without proof, the values of (i) $a^{12} \div a^2$; (ii) $(a^{12})^2$.

2. Simplify (i) $\frac{a}{2} - \frac{b}{2}$; (ii) $\frac{ab}{abc} - \frac{b}{2bc}$; (iii) $\left(\frac{1}{2} - \frac{1}{x}\right)2x$.

3. A man pays a bill of £x ys. with £$(x+1)$. How much change does he get?

4. Fig. 137 represents a rectangle. What is (i) its perimeter; (ii) its area?

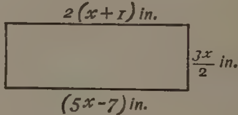


FIG. 137.

5. A man cycles uphill 10s miles from A to B at 6 miles an hour, and back again at 15 miles an hour. Find the time.

A. 9

1. Express in florins the difference between £x and $3x$ half-crowns.

2. Simplify (i) $10ab - ab$; (ii) $2x - \frac{x}{4}$; (iii) $x^2 - (x^2 - 1)$.

3. If $a = 3$, what are the values of
(i) $2a$; (ii) a^2; (iii) $3(a-2)$;
(iv) $2(a-3)$; (v) $a^2 - 4$; (vi) $(a-2)^2$?

4. A man walks in the direction $x°$ E. of N. along AB (see Fig. 138), he turns clockwise through $\left(\frac{x}{2}+5\right)$ degrees at B and walks along BC and then turns anti-clockwise through $(2x-15)$ degrees at C: he is now walking due North; find x.

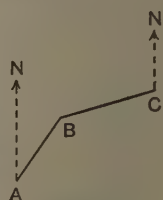


FIG. 138.

5. (i) A clock loses n minutes per week; express this loss in seconds per hour.
(ii) If a clock loses t seconds an hour, it loses T minutes a week; express t in terms of T.

A. 10

1. Three consecutive odd numbers are written down. Find their sum (i) if the greatest is G, (ii) if the least is g?

2. Simplify (i) $\frac{3}{11} \div 2$; (ii) $\frac{a}{b} \div c$; (iii) $\frac{2}{x} \div \frac{2}{y}$.

3. A cask of beer weighs W lb. when full, and w lb. when empty; find the total weight of n full casks and $2n$ half-full casks.

4. Simplify $x^2(x-2)+x(x^2+x-1)-2(x^3-2x^2)$.

5. A pyramid stands on an n-sided base. How many more edges than corners has it got? What is n if the sum of the number of edges and number of corners is 25?

A. 11

1. If $x=5$, $y=2$, show that x^2+y^2 lies between $(x+y)^2$ and $(x-y)^2$.

2. If $a=4$, $b=\frac{1}{2}$, find the value of (i) $2a^3$; (ii) $(2b)^3$; (iii) $3a^2b$; (iv) $a-\left(\frac{a}{2}+b\right)$; (v) $\frac{a}{b}$; (vi) $\frac{1}{a}+\frac{1}{b}$.

3. A walks v ft. per sec., B walks $(v-1)$ ft. per sec. They are standing 100 yd. apart; B walks away from A and A follows him. When will A overtake B?

4. For what value of l does $\frac{l}{5}$ exceed $\left(\frac{l}{2}-\frac{l}{3}\right)$ by 1?

5. Fig. 139 shows the lengths of the sides of a triangle in inches. Find x, if the triangle is isosceles; find also the lengths of the sides

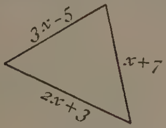


FIG. 139.

and ascertain whether the sum of any two sides is greater than the third. [There is more than one answer.]

A. 12

1. If x is an odd number, 24 is a factor of x^3-x. Show that this is true when $x=5$.

2. Write down (i) the square of $3ab^2$; (ii) a square root of $25a^4b^{16}$; (iii) the product of x^2-2x and $3x^2$.

3. The temperature rises at a steady rate from 8 a.m. till noon. At 8 a.m. it is 50° F., at 10 a.m. it is 56° F.; what is it at 11 a.m.? If at 8 a.m. it is x° F. and at 10 a.m. it is y° F., what is it (i) at noon; (ii) at 11.30 a.m.?

4. Simplify (i) $\frac{5abc}{5abc}$; (ii) $1+\frac{3a}{a}$; (iii) $\frac{2}{y}+\frac{2}{y}$; (iv) $\frac{t}{\frac{1}{2}}+\frac{t}{\frac{1}{4}}$.

5. A has £100, B has £50; but after B has paid A what he owes him, A has three times as much as B. What was the debt?

A. 13

1. From the formula $F=\dfrac{W(v-u)}{gt}$ find F, if $W=56$, $v=44$, $u=0$, $g=32$, $t=6$.

2. (i) Multiply $10a^2b^3$ by $3a^3b^2$ and divide the result by $6a^4b^4$. (ii) Simplify $18xz \div 42xyz$.

3. In Fig. 140, $\angle ACB = 4\angle ABC$ and CP bisects $\angle ACB$, find $\angle BPC$.

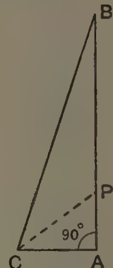


FIG. 140.

4. In Fig. 141, $AB=CD$; also $AD=2l$ in., $BC=2p$ in.; what is the length of AC ?

FIG. 141.

5. The base of a prism is an n-sided figure; what is the number of (i) its edges, (ii) its corners, (iii) its faces, including the two ends ?

What is n if the number of edges is $2\frac{1}{2}$ times the total number of faces ?

A. 14

1. What is the nth number in the set of numbers,

$$2\times 3,\ 3\times 4,\ 4\times 5,\ 5\times 6,\ \ldots ?$$

2. (i) Add together $\frac{1}{2}(y+z-x)$, $\frac{1}{2}(z+x-y)$, $\frac{1}{2}(x+y-z)$.

(ii) Simplify $x^2(x^2+x+1)-x(x^2+x-1)+(x^2-x+1)$.

3. Which is the greater, $\dfrac{a}{2}$ or $\dfrac{3a}{7}$?

Simplify

(i) $\dfrac{a^2}{b^2}\times ab$; (ii) $\dfrac{a+b}{2ab}-\dfrac{1}{a}$; (iii) $\dfrac{a^2}{b}\div\dfrac{b^2}{a}$.

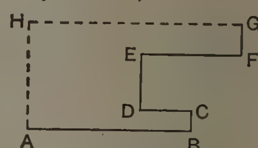


FIG. 142.

4. If $d+\pi d=1$, express d in terms of π.

5. In Fig. 142, all the corners are right-angled. The path $ABCDEFG$ is p in. longer than the path AHG. Can you find the length of one side of the figure ? If also the path $ABCDEF$ is q in. longer than the path $AHGF$, find the length of another side.

A. 15

1. (i) What is the value of the square of $\dfrac{x}{2}$ when $x=3$?

(ii) Divide the square of $3a^3b$ by the square of ab.

2. Tea costs $(2x+y)$ pence per lb. and coffee costs $(x+2y)$ pence per lb. Express by means of brackets the change in shillings out of £5, after paying for 20 lb. of tea and 18 lb. of coffee. Do not remove the brackets.

3. Express p shillings in the £ as a percentage.

4. A sentry starts from A and walks to and fro between A and B (see Fig. 143). How far is he from A when he has walked $(80n+15)$ yards, if n is (i) an even integer, (ii) an odd integer ?

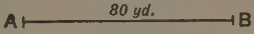


FIG. 143.

5. Fig. 144 shows the lengths of the sides of a rectangle in inches. What are they ?

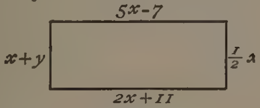


FIG. 144.

HARDER REVISION PAPERS. B. 1-10

B. 1

1. Write down the values of (i) $18a^5b \div 2ab$; (ii) a square root of $64x^{64}$; (iii) the H.C.F. of $6a^2$ and $6abc$; (iv) the L.C.M. of $2x^2$, $2y^2$ and $2xy$.

2. How many minutes are there between p minutes past 9 and q minutes to 12, on the same morning ?

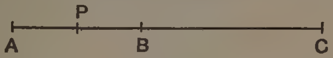


FIG. 145.

3. In Fig. 145, $AB=(R-r)$ inches, $AC=(2R+r)$ inches ; and P is the mid-point of AB ; what is the length of PC ?
If $R=2r$, prove that $BC=4AB$.

4. Fill in the gaps in the following table, where $4x+3y=10$.

$x=0$	—	1	—	2	—
$y=$—	0	—	1	—	3

5. A man's ordinary wage is 9d. per hour : overtime pay is 1s. per hour. A man works 54 hours one week and earns 42s.; how many hours did he work overtime ?

B. 2

1. The area of a triangle is

$$\tfrac{1}{4}\sqrt{\{(a+b+c)(b+c-a)(c+a-b)(a+b-c)\}} \text{ sq. in.,}$$

if the sides are of lengths a in., b in., c in.; find the area of a triangle whose sides are (i) 5 in., 5 in., 6 in.; (ii) l in., l in., l in.

2. Add together $\dfrac{2a+b}{4}$, $\dfrac{a-2b}{12}$ and $\dfrac{2b-a}{3}$.

3. If $a=2x-y$, $b=x-2y$, prove that $a^2+b^2=65$ when $x=5$, $y=2$.

4. A boy buys 10 dozen papers at 8 pence a dozen; he sells some of them at one penny each and returns the rest, getting 7 pence a dozen for them. He gains half-a-crown. How many does he sell?

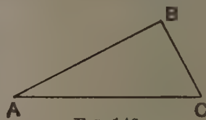


FIG. 146.

5. In Fig. 146, $AC+BC$ is twice AB and $AB+AC$ is three times BC. The perimeter is 2 ft. Find the length of each side.

B. 3

1. Under the Derby Scheme of 1915, men of the same number of years of age were placed in the same group; *single* men of ages from 18 to 40 were placed in groups numbered from 1 to 23 respectively; *married* men of ages from 18 to 40 were placed in groups numbered from 24 to 46 respectively. In what group was a man of age A years placed, (i) if single, (ii) if married?

2. If $x=3$, $y=1$, find the value of

$$\text{(i) } xy-x\text{; (ii) } (x+y)^2\text{; (iii) } x^2+y^2\text{; (iv) } \frac{x+y}{x+3y}.$$

3. A man buys a dog for £5 and sells it for £$(5+g)$. What is his gain per cent.?

4. The external dimensions of a closed wooden box are $12a$ in., $10a$ in., $8a$ in.; the wood is a in. thick. Find (i) the area of the outside, (ii) the area of the inside, (iii) the volume of wood used in making the box.

5. A jug and basin together cost p shillings; the jug costs half a crown more than the basin; find the cost of each.

B. 4

1. Simplify $a(a-b+c)+b(a+b-c)+c(b+c-a)$, and then express the result in terms of a, if $a=2b=\frac{2}{3}c$.

2. How much per cent. is w cwt. of w tons?

3. Solve (i) $\frac{7}{t}-3=\frac{5}{t}+2$; (ii) $\frac{1}{2z}=\frac{2}{z+2}$.

4. The relation between the thickness T inches and the diameter D inches of a steam engine cylinder is

$$T=\frac{\sqrt{D}}{5}+\cdot015D.$$

What is T when $D=16$? What is the increase in T if D is now increased by 9?

5. Find in terms of a the perimeter of a rectangle x in. long, y in. broad, if $\frac{x}{2}=\frac{y}{3}=\frac{a}{5}$.

B. 5

1. The value of silver money is proportional to its weight; 1 lb. of silver is made into 66 shillings. What should be the weight of £p worth of silver?

2. Simplify (i) $\frac{p}{p+q}+\frac{q}{p+q}$; (ii) $\frac{p^2}{q}\times\frac{q^2}{p}+pq\left(\frac{p}{q}+\frac{q}{p}\right)$.

3. If the xth of September is a Tuesday, what day of the week is the $(x+1)$th of October?

4. Solve $\frac{x+1}{3}-\frac{x+2}{4}=\frac{x+3}{5}-\frac{x+4}{6}$.

5. Three-quarters of an even number equals two-thirds of the next even number. Find the numbers.

B. 6

1. Find a formula which includes the following statements:

(i) 35s. per cwt. is $\left(\frac{3+5+\frac{5}{2}}{14}+3\right)$ pence per lb.;

(ii) 67s. per cwt. is $\left(\frac{6+7+\frac{7}{2}}{14}+6\right)$ pence per lb.;

(iii) 94s. per cwt. is $\left(\frac{9+4+\frac{4}{2}}{14}+9\right)$ pence per lb.

Prove your formula.

2. In a tennis match between A and B, the scores in games of the three sets are:

A	6	$3x-2y$	6
B	$x+y$	6	y

By how many games does A defeat B?

3. Simplify $\left(\frac{x^{12}}{x^3}\right)^3\div\left(\frac{x^6}{x^3}\right)^3$.

4. (i) A road rises steadily n feet per mile of road; how many inches does it rise in 100 yards?

(ii) What is the perimeter of a rectangle of area A sq. in. and length l in.?

5. If the pth day of January and the $(2p)$th day of February are both Sundays, find p.

B. 7

1. The fine for returning a book late to a library is half-a-crown for any time during the first week and one shilling a day afterwards. What is the fine on a book returned n days late, $(n>7)$?

2. Simplify $x(y-1)+y(x-1)+xy\left(\frac{1}{x}+\frac{1}{y}\right)$.

3. Through how many degrees does the hour hand of a clock turn between t minutes past six and t minutes to seven? Evaluate the expression (i) if $t=15$, (ii) if $t=30$.

4. Find the value of $a+b$ if $\frac{1}{2}=\frac{1}{3}+\frac{1}{a}$ and $\frac{1}{3}=\frac{1}{4}+\frac{1}{b}$.

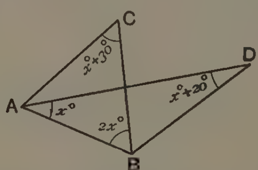


FIG. 147.

5. In Fig. 147, find x if $\angle CBD=2\angle CAD$.

B. 8

1. In using a screw-driver I find that k complete turns are needed to drive a screw through a 2-inch plank. Through what distance does the screw head advance when the screw-driver is turned through 120°?

2. Simplify $\left(\frac{a}{2}+\frac{3}{4}a-a\right)\div\frac{a}{2}$.

3. Fill in the blanks in

(i) $\frac{1}{a}=\frac{a}{\ }$; (ii) $x=\frac{\ }{\sqrt{4x^2}}$; (iii) $\frac{a^2-1}{a}=\frac{\ }{2}-\frac{1}{a}$.

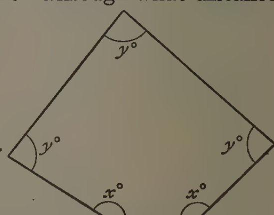


FIG. 148.

4. The sum of the edges of a cube is l feet; what is (i) the volume in cu. in., (ii) the area of the surface in sq. in.

5. With the data of Fig. 148, find a formula for x in terms of y.

B. 9

1. The rates for inland letter post are: not more than 2 oz., $1\frac{1}{2}$d. and for each additional 2 oz. or fraction thereof, $\frac{1}{2}$d. Find the cost of postage of a letter weighing W oz., where W is an odd integer greater than unity.

2. Simplify (i) $\frac{x}{\frac{2}{3}} - \frac{3x}{2\frac{1}{2}}$; (ii) $\frac{x}{\frac{2}{3}} \div \frac{3x}{2\frac{1}{2}}$.

3. Find the L.C.M. of 12, $2a^2$, $9a^3b$, ab^2. What is the least expression this L.C.M. must be multiplied by, in order to make the result (i) a perfect square, (ii) a perfect cube?

4. If $x = a + 2b$, $y = 3a - b$ and $a = 2b = 6$, prove that $\frac{x}{3} - \frac{y}{5} = 1$.

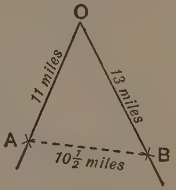


FIG. 149.

5. As a rule, P bicycles twice as fast as Q walks. Each leaves A at 2 p.m., they agree to meet at B (see Fig. 149). P follows the roads AO, OB, Q goes straight across country. Owing to the wind, their speeds are each one mile an hour less than usual. They arrive at B at the same time. What was the time of arrival?

B. 10

1. Subtract (i) $2a - 3(b - c)$ from $5a + (b + c)$; (iii) $3x^2$ from $(3x)^2$.

2. Simplify (i) $\frac{15xy}{12y^2} \div \frac{25x^3}{2xy}$; (ii) $x - \frac{x}{100}$.

3. A box is l ft. long, b ft. wide, h ft. high; what length (in feet) of cord is needed to bind it once round in each of the three directions, allowing 2 ft. extra for knots?

4. Solve (i) $4x = 5x$; (ii) $\frac{12}{x} = 96$.

Can you find numerical values of x and y for which

$$x + 2 = 2y = 5 - x\ ?$$

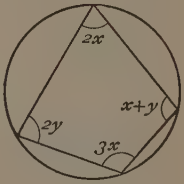


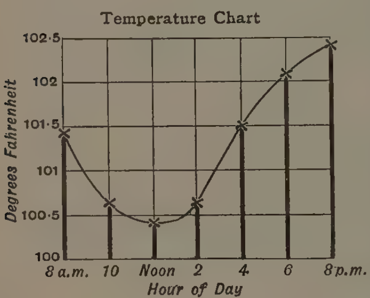
FIG. 150.

5. Fig. 150 shows the angles of a cyclic quadrilateral, in degrees; find the values of x and y. [The opposite angles of a cyclic quadrilateral are supplementary.]

CHAPTER VI

GRAPHS OF CONTINUOUS FUNCTIONS

In Chapter V. the reader was asked to consider, in the case of each column-graph that was drawn, whether meanings could be attached to intermediate uprights. Sometimes this was possible, sometimes impossible. Suppose, for example, successive uprights represent the temperatures of a fever-patient at stated intervals throughout the day. If the intervals are sufficiently frequent, the uprights will give a good idea of how the patient's temperature changes during the day, and, if we insert by eye intermediate uprights, these will probably represent with fair accuracy the temperatures at the intermediate times. The temperature is a continuous function of the time, although of course there may be comparatively abrupt changes of temperature at certain times. But the temperature chart may be taken as an example of continuous variation, and we call the (smooth) curve which passes through the top points of the uprights a **line-graph** representing the relation between the temperature and the time.



Note. If possible, the graphs drawn by each pupil throughout this chapter should be preserved, so that they can be used at the end of the chapter for summarising the various aspects of graphical work and their bearing on functionality.

The whole of the exercises in this chapter should not be worked out at any one time, and the work of the chapter generally may be taken concurrently with that of Chapter VII. At a first reading, the object should be to enable the pupil to draw the inferences at the end of the chapter from the exercises that have been done individually or orally, and so be prepared for the work of Chapter VIII.

Line-Graphs, Continuous Variation.

Example I. The height of the barometer in inches is recorded at hourly intervals on a certain day as follows :

Time -	9 a.m.	10 a.m.	11 a.m.	12	1 p.m.	2 p.m.	3 p.m.
Height in inches	29·55	29·70	29·77	29·70	29·90	29·72	29·15

Represent these readings graphically.

All readings lie between 29 in. and 30 in.; we therefore take 29 in. as our origin on the axis up the paper. The unit selected for each axis is shown in the figure.

Each observation can now be represented by a point, marked by a cross, on the squared paper.

Now suppose that an automatic recording machine has been employed; then the pointer of the machine will have marked in, not only the isolated points corresponding to the given readings,

BAROGRAPH.

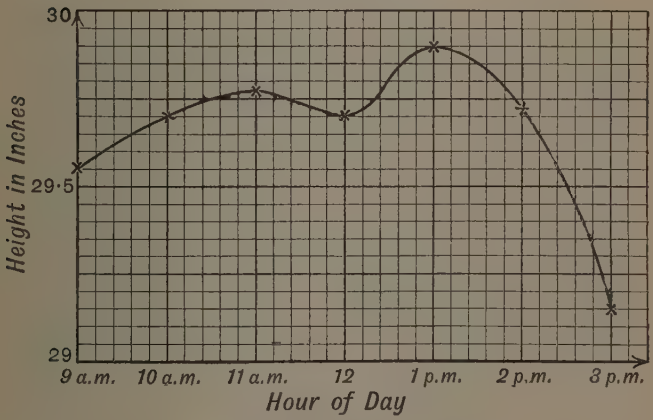


FIG. 151.

but also a *continuous curve*, passing through them. We therefore proceed to draw free-hand a smooth curve through the isolated points and we say that each point on this curve represents the height of the barometer at the time to which it corresponds.

Oral work, using Fig. 151.

(i) What is approximately the height of the barometer at 9.36 a.m., 12.24 p.m., 1.24 p.m., 2.48 p.m. ?

(ii) At what times is the height of the barometer 29·65 in., 29·80 in., 29·45 in. ?

(iii) Between what times was the barometer rising ?

(iv) Between what times was the barometer above 29·65 in. ?

(v) How much did the barometer fall between 1.30 p.m. and 2.30 p.m. ?

(vi) How much did the barometer rise between 9.30 a.m. and 10.30 a.m. ?

(vii) What inference can you draw from noticing that a special part of the graph slopes *downwards* or from noticing that one part slopes downwards *more steeply* than another part ?

Exercise VI. *a*.

1. Fig. 152 shows the travel graphs of two people. The graph *OABCDEFG* corresponds to a man who successively

TRAVEL GRAPHS.

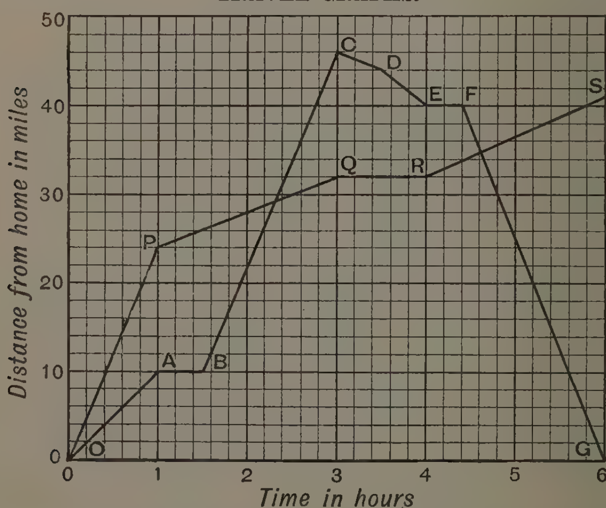


Fig. 152.

cycles, motors, walks, takes a bus and a train : the graph *OPQRS* corresponds to a man who motors and then walks.

Both men leave home at 10 a.m.

(i) Take the graph *OABCDEFG* and describe the journey in detail and calculate the various speeds of travel.

(*a*) Can you tell at a glance when he is moving fastest and when slowest ?

(*b*) How far does he go altogether ?

(*c*) At what time does he start on his return journey ?

(*d*) What would be the difference of meaning if parts of the graph were curved instead of being straight ?

(ii) Repeat (i) for the graph *OPQRS*.

(iii) If they pass one another, where and when does it happen?

2. A motorist leaves home at 9 a.m. ; the mileage recorded by his cyclometer, originally set at zero, is as follows :

Time -	9.10	9.20	9.25	9.30	9.40	9.50	10.0	10.10	10.20
Mileage	2	5	9	12	16	19	24	31	37

Draw his travel graph.

(i) What is the reading at 9.36, 9.54, 10.15 a.m. ?

(ii) How far does he travel between 9.35 a.m. and 10.5 a.m. ?

(iii) What is the time when he has travelled 14 miles, 21 miles, 33 miles ?

(iv) What is his approximate speed at 9.40 a.m., 10.10 a.m. ?

3. Interpret the travel graph in Fig. 153, stating the different speeds. What is the average speed for the whole journey ?

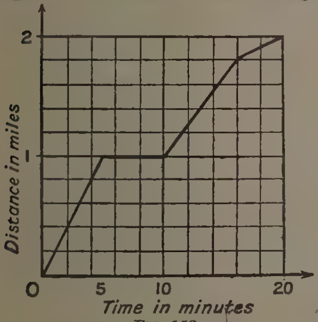


FIG. 153.

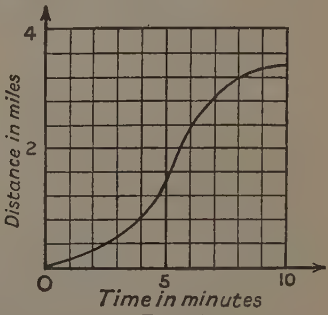


FIG. 154.

4. Draw on an enlarged scale the travel graph in Fig. 154. State the total distance travelled in the first 4 minutes, the

first 6 minutes, the first 8 minutes. Find approximately, in miles per hour, the average speed for the first 4 minutes and the speed after 3 minutes and after 8 minutes. After what time is the speed greatest ?

5. Draw a travel graph for the following extract from Bradshaw, with the time-axis across the page.

	London.	Bath.	Bristol.	Weston.	Bridgwater.
Miles	0	107	118	136	151
Arrival -	—	2.53	3.17	4.0	4.40 p.m.
Departure	1 p.m.	2.59	3.30	4.10	—

Which part of the graph makes the greatest angle with the time-axis, and what does this mean ? Find graphically the average speed for the whole journey.

6. The amount of water in a tank t seconds after the escape pipe is opened is n gallons where t, n are related as follows :

t - -	0	5	10	15	20	30	40	50	60	70
n - -	64	56·2	49	42·3	36	25	16	9	4	1

Represent graphically the relation between n and t.

How many gallons run out in the first 25 seconds and in the first 55 seconds ?

Find the rate in gallons per sec. at which the water is escaping after (i) 15 sec., (ii) 30 sec.

7. The following are the Census returns for England and Wales :

Year - -	1861	1871	1881	1891	1901	1911	1921
Population in millions -	20·07	22·71	25·97	29·00	32·53	36·07	37·88

Represent this table by a graph.

What was the probable population in 1895 and 1906 ?

What was the annual rate of increase in 1901 ?

Is the return for 1921 surprising ?

8. Greenwich time for Sunset in 1927 is as follows:

	Ap. 1.	May 1.	May 31.	June 30.	July 30.	Aug. 29.	Sept. 28.
p.m.	6.30	7.20	8.3	8.19	7.51	6.54	5.45

Represent this table by a graph.

What is the time of sunset on April 16, June 8, Sept. 20 ?

On what dates does the sun set at 6.45 p.m., 7.43 p.m. ?

9. If £1 is allowed to accumulate at 4 per cent. per annum compound interest, the amount is as follows:

Number of years	0	5	10	20	30	35
Amount	1	1·22	1·48	2·19	3·24	3·95

Represent this table by a graph.

What is the amount after (1) 15 years, (ii) 25 years, (iii) 33 years ?

10. The following table shows the altitude of the Sun and the length of the shadow of a vertical pole 10 ft. high in London at noon on the first day of each month in 1927.

	Jan 1.	Feb. 1.	Mar. 1.	Ap. 1.	May 1.	June 1.
Altitudes in degrees	15·4	21·2	30·6	42·8	53·4	60·5
Length in feet	36·3	25·8	16·9	10·8	7·43	5·66

	July 1.	Aug. 1.	Sept. 1.	Oct. 1.	Nov. 1.	Dec. 1.
Altitude in degrees	61·7	56·7	47·1	35·6	24·3	16·8
Length in feet	5·38	6·57	9·29	14·0	22·1	33·1

Represent these results graphically.

When was the altitude (i) 27°, (ii) 45° ?

When was the length of the shadow (i) 20 ft., (ii) 10 ft., (iii) 8 ft.?

At what times of year does the altitude (*a*) increase most rapidly, (*b*) decrease most rapidly, (*c*) remain nearly stationary ?

11. The following table shows the expectation of life of an Englishman at different ages :

Age in years	30	40	50	60	70	80	90
Expectation in years	33·2	26·5	19·9	13·6	8·6	5·2	2·8

Represent this table by a graph.

(i) What is the expectation of life at the age of 34, 53, 66 ?

(ii) At what age is the expectation of life 22, 16, 11 ?

12. Use the data of No. 11 to draw a graph showing the relation between present age and probable length of life.

What is the probable length of life at the age of 45, 75 ?

In the above examples, the independent variable in each case has been the time ; the following examples exhibit other types of variation.

13. A spiral spring is suspended from one end and its length is measured when different weights are attached to the other end.

Weight in gr.	10	15	30	50	75
Length in cm.	22	24	30	38	48

Represent graphically the relation between the length and the load.

What is the length if the load is 20 gr., 40 gr., 65 gr. ?

What is the load if the length is 23 cm., 28 cm., 42 cm. ?

What is its natural length, *i.e.* the length when there is no load ?

Is the graph a straight line ? If so, what does this mean ?

How can you interpret the slope of the line ?

14. The British amateur running records are as follows :

Distance in yd.	150	200	440	600	880	1000
Time in sec.	14·6	19·4	47·0	70·8	112·2	132·4

Represent this table by a graph.

What would be the probable record for 500 yd., 750 yd. ?

The American record for 300 metres (1 m. = 1·09 yd.) is 33·2 sec. ; how does this compare with British records ?

Should this graph be a straight line ?

15. The time of a complete oscillation for pendulums of different lengths (in London) is as follows:

Length in ft.	1	2	3	4	5	6
Time in sec.	1·11	1·57	1·92	2·21	2·48	2·71

Represent this table by a graph.

What should be the length if the time is to be 2 sec.?

What is the time if the length is 2·5 ft., $4\frac{3}{4}$ ft.?

The pendulum of a clock should make complete oscillations every two seconds, if the clock is keeping time; how should the length be corrected if a complete oscillation takes 2·1 seconds?

What alteration in length would be required to reduce the time of oscillation from 1·3 sec. to 1·2 sec.; is it the same as before?

16. The following table gives the distance, d yards, in which a train running at V miles per hour can be stopped.

V	30	40	45	50	60
d	100	178	225	278	400

Represent this table by a graph.

(i) How much further will a train run if the brakes are put hard on, when the speed is 35 m.p.h., 55 m.p.h.?

(ii) How fast is a train travelling if it can be stopped in 200 yards?

(iii) Compare the *extra* distance that must be allowed, for an increase of velocity of 1 mile an hour, in the case of two trains, one travelling at 35 m.p.h. and the other at 55 m.p.h.?

17. When erecting a hoarding in an exposed place, it is necessary to take account of the pressure exerted by the wind. If the wind is blowing at v miles per hour, it exerts a pressure of P lb. per sq. ft. given by the following table:

v	5	10	15	20	25
P	0·12	0·51	1·15	1·88	2·95

Represent this table by a graph.

What pressure in lb. per sq. ft. is caused by a wind blowing at 12 m.p.h., 22 m.p.h.?

What is the velocity if the pressure in lb. per sq. ft. is 0·4, 1·6?

The surface area of a hoarding is 360 sq. ft.; what total pressure must it be able to withstand in a gale blowing at 23 m.p.h.?

18. A jet of water from a $\frac{3}{4}$-inch nozzle under a pressure of P lb. per sq. in. attains an effective height h ft. where P, h are connected as follows:

P	40	50	60	80	90	100
h	61	67	72	79	81	83

Represent this table by a graph.

(i) What is the effective height for a pressure of 70 lb. per sq. in.?

(ii) What pressure is required to attain an effective height of 70 feet?

(iii) Compare the increase of effective height per unit increase of pressure in lb. per sq. in. for a pressure of 45 lb. per sq. in. and for a pressure of 95 lb. per sq. in.

Graphs as Ready Reckoners.

Example II. Draw a graph which shows the squares of numbers from 0 to 10, and the square roots of numbers from 0 to 100.

We have the following table of values connecting a number N and its square.

N	0	1	2	3	4	5	6	7	8	9	10
N^2	0	1	4	9	16	25	36	49	64	81	100

For the scale of numbers, let 5 small divisions represent 1, and for the scale of squares, let 5 small divisions represent 10.

When we have plotted the values in the above table, we find that the plotted points are rather far apart when the number is large; to make the drawing of the graph easier, we therefore take some additional values:

N	7·5	8·5	9·5
N^2	56·25	72·25	90·25

We now draw a smooth curve through the plotted points and obtain the graph in Fig. 155.

SQUARES AND SQUARE ROOTS.

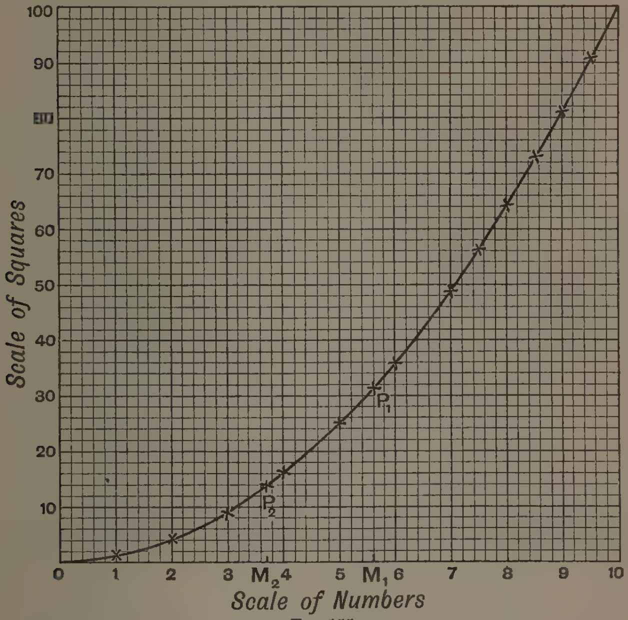


FIG. 155.

If now we go any distance, say N units, along the scale-of-numbers axis, the distance up on to the graph represents the value of N^2. Thus, if we go 5·6 units along the scale-of-numbers axis we arrive at M_1, and the distance M_1P_1 up on to the curve represents $(5{\cdot}6)^2$. Therefore we have from the graph $(5{\cdot}6)^2 \simeq 31{\cdot}4$.

Conversely, consider the point P_2 on the graph where $M_2P_2 = 14$. To arrive at M_2, we must go approximately 3·73 units along the scale-of-numbers axis ; $\therefore\ (3{\cdot}73)^2 \simeq 14$ and $\therefore\ \sqrt{14} \simeq 3{\cdot}73$.

This shows that if we label the axis up the paper "The Scale of Numbers," then the axis across the paper becomes "The Scale of Square-roots."

Oral work on Fig. 155.

(i) Read off the squares of 6·8, 8·2, 9·8.

(ii) Read off the square roots of 60, 20, 86, 43.

EXERCISE VI. *b*.

1. Draw a graph from which you can read off in ft. per sec. a speed given in miles per hour, for speeds up to 60 miles an hour.

Miles per hour -	0	15	30	45	60
Feet per sec. -	0	22	44	66	88

From the graph,

(i) express in ft. per sec.: 12, 26, 48, 53 m.p.h.;

(ii) express in m.p.h.: 20, 31, 50, 67 ft. per sec.

2. Draw a graph which shows the cubes of numbers from 0 to 5.

Read off from the graph, (i) the cubes of 2·3, 3·8, 4·7; (ii) the cube roots of 20, 36, 74.

For what range of values of N is N^3 less than 50?

3. A long straight lever is balanced about its centre O (see Fig. 156), a body of weight 10 lb. is suspended from a point

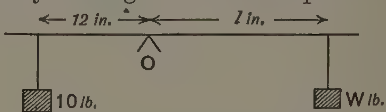


FIG. 156.

12 in. from O; the balance is maintained by a body of weight W lb. suspended from a point l in. from O.

By experiment (or by using the law of the lever), we find that W and l are connected as follows:

W	5	10	15	20	30	40
l	24	12	8	6	4	3

Represent these results graphically.

(i) Read off the values of l for $W=14$, 24, 35.

(ii) Read off the values of W for $l=9$, 16, 21.

4. The relation between the area of a circle and its diameter is shown in the following table:

Diameter in inches -	1	2	3	3·5	4	4·5	5
Area in sq. in. -	0·79	3·14	7·07	9·62	12·57	15·90	19·63

Represent these results graphically.

What is the area of a circle of diameter 1·7, 3·2, 4·8 in. ?

What is the diameter of a circle of area 5·3, 11·3, 14·5 sq. in.?

5. The relation between the girth of a circular cylinder and the area of its cross-section is as follows :

Girth in inches -	2	4	6	7	8	9	10
Area in sq. in. -	0·32	1·27	2·86	3·90	5·09	6·45	7·96

Represent these results graphically.

(i) What is the area of the cross-section if the girth is 4·7, 7·3, 8·6 in. ?

(ii) What is the girth if the area of the cross-section is 3, 6, 7 sq. in. ?

6. Draw a ready-reckoner graph for converting degrees Centigrade to degrees Fahrenheit, from the relation given below :

Centigrade -	0	50	100
Fahrenheit -	32	122	212

(i) Express in Fahrenheit : 35° C., 60° C., 73° C.

(ii) Express in Centigrade : 98° F., 150° F., 180° F.

7. The marks obtained on an examination paper run from 23 to 87 : draw a ready-reckoner graph for converting them so as to run from 0 to 100.

(i) Write down the scaled mark, if the original mark is 36, 50, 72.

(ii) Write down the original mark, if the scaled mark is 10, 44, 91.

(iii) What mark is unchanged by the conversion ?

8. Make a table showing the relation between a number and its reciprocal for numbers from 0·2 to 10 (the reciprocal of 4 is $\frac{1}{4}$ = 0·25) and then construct a ready-reckoner graph. Write down the reciprocals of 1·3, 4·8, 6·7.

9. The breaking strain of a flexible steel wire rope was tested with the following results :

Circumference in inches -	1·5	2	2·5	3	3·5	4
Breaking strain in tons -	4·0	7·0	12	18	26	33

(i) What is the breaking strain if the circumference is 2·8, 3·7 in.?

(ii) What length of circumference is required for a breaking strain of 9, 20 tons?

10. Experiments with a screw-jack showed that the load to be raised and the necessary effort were related as follows:

Load in lb.-wt. -	100	120	160	180	200
Effort in lb.-wt. -	12·0	13·8	17·6	19·6	21·5

(i) What is the effort necessary if the load is 140, 170, 186 lb.-wt.?

(ii) What load can be raised if the effort is 13, 16, 20 lb.-wt.?

11. The height h ft. above sea-level of a place may be found by observing the temperature $T°$ Fahrenheit at which water boils: the relation is as follows:

T	212	210	205	200	195	192
h	0	1050	3700	6400	9150	11,000

(i) What is the height if the temperature is 207° F., 202° F., 193° F.?

(ii) What is the temperature corresponding to a height of 5000 ft., 8000 ft.?

12. Draw a graph to show the amount of tax paid on incomes up to £800 if (i) no tax is paid on the first £135, (ii) tax is paid at the rate of 2s. 6d. in the £ on the next £225 of a man's income, (iii) tax is paid at the rate of 5s. in the £ on the rest. (i) Find from your graph the tax if the income is £340, £460, £710. (ii) What is the income of a man who pays £120 tax?

Graphs of Given Functions.

In most of the examples already considered, the data for a table of values have been obtained by observation or experiment. It often happens, however, that the law governing the variation of the quantities under discussion can be expressed as a formula from which the table of values can be deduced.

Example III. At one boarding house, the charge is 10 shillings down and 6 shillings a day. At a second boarding house, the charge is 7s. 6d. a day without any initial payment.

If it costs C shillings to stay n days, express C in terms of n in each case and represent the relations between C and n graphically.

At the first boarding house, the charge for n days is $6n$ shillings in addition to an initial payment of 10 shillings; $\therefore$ the total charge is $10+6n$ shillings;

$$\therefore\ C=10+6n.$$

At the second boarding house, the charge for n days is $7\frac{1}{2}n$ shillings;

$$\therefore\ C=7\tfrac{1}{2}n.$$

By substituting in these formulae we obtain the necessary table of values:

n	0	2	5	8	10
$10+6n$	10	22	40	58	70
$7\frac{1}{2}n$	0	15	37·5	60	75

Plotting them we obtain the graph BD for the cost of living at the first boarding house and the graph OA for that at the second boarding house.

BOARDING-HOUSE CHARGES.

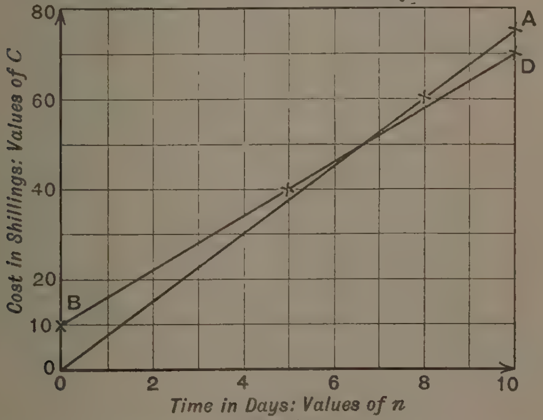


FIG. 157.

The graph BD represents the various values of the expression or function $10+6n$ when n varies from 0 to 10. We therefore say that BD is the graph of the function $10+6n$; any expression whose value depends on the value of n is called a **function of n**, and n is

called the **independent variable** of the function. Similarly, OA is the graph of the function $7\frac{1}{2}n$.

Oral work on Fig. 157.

(i) After how many days' stay does the second boarding house cost more than the first ?

(ii) The bill at either place increases each day by the same amount ; how does this fact show itself in the graphs ?

(iii) The slope of OA is greater than the slope of BD ; what does this mean ?

(iv) What would be the effect on the graph if the first boarding house suddenly raised its charge to 9s. per day ?

(v) What is the degree of each of the functions of n represented ?

Exercise VI. *c*.

1. A boy starts the term with 5 shillings and spends 4d. a day. Draw a graph to show how his balance diminishes. If his balance is b shillings after n days, express b as a function of n. How does the fact that his balance diminishes by a constant amount each day show itself in the graph ?

With the same axes, draw a graph showing how his balance would diminish if he spent 6d. a day. Which graph has the greater slope and why ?

What is the degree of each function ?

2. The unstretched length of a spiral spring is 8 in. It is suspended from one end and bodies of various weights are attached to the other end. Each increase of 4 oz. in the weight of the body attached causes an extra extension of 1 in. What relation connects the total length l in. of the spring with the weight W lb. of the attached body. Draw a graph to show the total length for bodies up to 3 lb. wt. What function is represented by this graph ? Would the slope of the graph be increased or diminished if a less elastic spring had been taken ?

3. The charge for x lb. of luggage for a certain journey is $\left(\frac{x}{2}-15\right)$ pence. Draw a graph to show the scale of charges from 40 lb. to 100 lb. Read off from the graph (i) the charge for 56 lb. ; (ii) the weight for which the charge is 2 shillings. Use a ruler to find how much free luggage is allowed.

4. If $F°$ Fahrenheit is the same as $C°$ Centigrade, then $F=\frac{9}{5}C+32$. Draw a graph for converting degrees Centigrade to degrees Fahrenheit for values of C from 0 to 25. Express

8·5° Centigrade in degrees Fahrenheit and 60° Fahrenheit in degrees Centigrade.

5. A train is travelling at 50 miles an hour when the brakes are put on; these reduce its speed by 3 miles an hour every 4 seconds. Find an expression for its speed, v miles an hour, when the brakes have been in action for t seconds. Represent this function of t graphically. Find from the graph how long it is before the speed is reduced to 20 m.p.h. and before the train stops.

What is the effect on the graph if more powerful brakes are employed?

6. A man's rate of pay is as follows: ordinary time, 1s. 6d. an hour; overtime, 2s. an hour. The ordinary working day is 7 hours. If he receives P shillings for working t hours one day, express P as a function of t, (i) if $t<7$, (ii) if $t>7$. Represent graphically a day's wages for values of t from 0 to 10.

What is the meaning of the change of slope? How long does he work if he receives 14s. for a day's work?

7. A's salary is £200 a year for his first year and he receives annually an increase of £15 a year; B's salary is £150 a year for his first year and he receives annually an increase of £20 a year. Draw graphs showing their salaries for the first twelve years.

What functions are represented by these graphs? After what time does B's salary exceed A's?

8. When a railway cutting, h ft. high, is faced with a brick wall, the thickness of the wall at its base is t in., where

$$t=\frac{h}{3}+3 \text{ if } h<18 \quad \text{and} \quad t=\frac{2h}{3}-3 \text{ if } h>18.$$

Represent graphically values of t for values of h from 9 to 30. What is the meaning of the change of slope?

9. Draw with the same scale and axes the graphs of $5-\frac{x}{2}$ and $2+\frac{x}{3}$ for values of x from 0 to 6. For what value of x is $5-\frac{x}{2}$ equal to $2+\frac{x}{3}$?

10. Without writing down a table of values, sketch roughly the graphs of (i) $2x$, (ii) $2x+1$, (iii) $4x$, (iv) $10-x$, (v) $10-2x$.

Example IV. The manager of a shop finds that the profits per week depend on how he divides his time between the shop and the office. If he spends n hours per day in the shop, the weekly

profits £p are given by the relation $p = 11 + 24n - 3n^2$. Draw a graph showing how the value of p varies for values of n from 0 to 8.

First make a table of values for the function $11 + 24n - 3n^2$. This is best done by writing down the row corresponding to $24n$, then the row corresponding to $3n^2$; subtracting, we obtain the row corresponding to $24n - 3n^2$; then write down the row for

$$11 + 24n - 3n^2.$$

The work should therefore be arranged as under :

n - - -	0	1	2	3	4	5	6	7	8
$24n$ - -	0	24	48	72	96	120	144	168	192
$3n^2$ - -	0	3	12	27	48	75	108	147	192
$24n - 3n^2$ -	0	21	36	45	48	45	36	21	0
$11 + 24n - 3n^2$	11	32	47	56	59	56	47	32	11

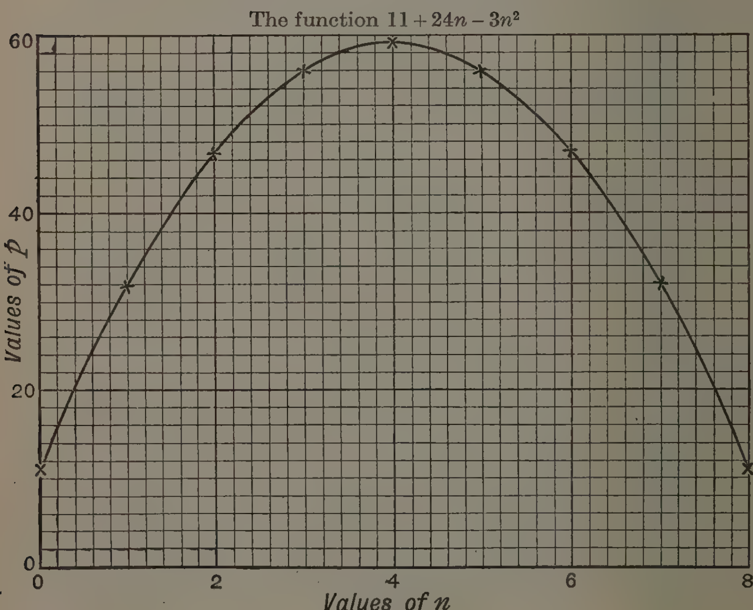


FIG. 158.

Plotting these results and drawing a smooth curve through the points obtained, we have Fig. 158.

Note. If fractions appear in the table of values, *express them as decimals*, because it makes the plotting easier.

Oral work on Fig. 158.

(i) Is it more profitable for the manager to be $2\frac{1}{2}$ hours or $3\frac{1}{2}$ hours in the shop ? How does this show itself in the graph ?

(ii) What is the most profitable time for him to remain in the shop ?

(iii) Is $2\frac{3}{4}$ hours or $5\frac{3}{4}$ hours a better time for being in the shop ?

(iv) Is the slope of the graph where $n=1$ greater or less than the slope where $n=3$, and what does this mean ?

(v) What is the profit if he stays in the shop $1\frac{1}{2}$ hours, $5\frac{1}{2}$ hours ?

(vi) How long is he in the shop if the profit is £20, £40, £50 ?

(vii) Does this graph represent direct proportion ? Is the profit proportional to n, the number of hours spent in the shop ?

(viii) What is the degree of the function, whose graph has been drawn ?

Exercise VI. *d*.

After each example in this exercise, answer the following questions :

(i) *Is the graph of the function a straight line ?*

(ii) *Do the quantities vary in direct proportion ?*

(iii) *Do the quantities vary in inverse proportion ?*

(iv) *Can you say what the degree of the function, represented by the graph, is ?*

1. A leaden sheet 30 inches wide is bent to form a gutter of rectangular section as shown in Fig. 159. Make a table

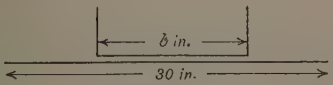


FIG. 159.

showing the relation between the area of the cross-section, A sq. in. and the width, b in., of the gutter. Represent this graphically. What width gives the maximum area for the cross-section ?

Express A as a function of b.

2. A stone projected vertically upwards with velocity 40 feet per sec., is s ft. above its starting point after t sec., where $s=40t-16t^2$. [This relation neglects air-resistance.]

Represent graphically this relation between s and t, for values of t from 0 to $2\frac{1}{2}$.

Find from the graph (i) how long it takes to reach its highest point, (ii) how long it takes to come down again to the ground, (iii) at what time it is 16 ft. above the ground.

How does the slope of the graph alter, and what does this mean?

3. A marble is projected down a gentle slope and travels s feet in t sec. where $s=8t+\frac{1}{2}t^2$. Represent graphically the relation between s and t for the first 5 seconds. How long does it take to travel 35 feet? How does the slope of the graph alter, and what does this mean?

4. In Fig. 160, $ABCD$ is a rectangle, $AB=10$ in., $BC=8$ in.; $AP=AS=CQ=CR$. If $AP=2$ in., find the area of each of the four triangles at the corners and so find the area of $PQRS$.

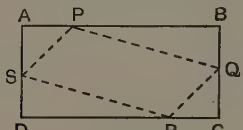


FIG. 160.

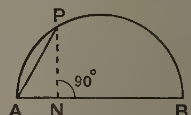


FIG. 161.

Make a table of values showing how the area of $PQRS$ varies with the length of AP, as AP increases from 0 to 8 in. Represent this relation graphically and find from the graph the maximum area of $PQRS$.

5. The length of a degree of longitude in latitude $x°$ is approximately $\left(69-\frac{x^2}{100}\right)$ miles. Represent this function graphically for values of x from 0 to 60. What is the length of a degree of longitude through London (lat. $51\frac{1}{2}°$)? In what latitude is a degree of longitude 63 miles long?

6. In Fig. 161, AB is the diameter of a circle of radius 5 cm. Draw an accurate figure and make the necessary measurements for obtaining a table of values connecting the length of AN and AP as P moves along the semicircle. Represent this relation, graphically.

Find from your graph the length of AN when $AP=2AN$.

7. With the data of No. 6, make the necessary measurements for obtaining a table of values connecting the lengths of AN and the area of $\triangle ANP$. Construct a graph and find the length of AN for which the area of $\triangle ANP$ is greatest.

8. In Fig. 162, $ABCD$ is a square of side 10 in.; there are equal squares, sides AP and BQ, at the corners. What is the

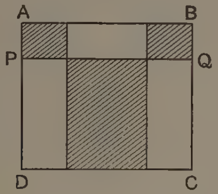


FIG. 162.

shaded area if AP is (i) 1 in., (ii) 2 in.? Make a table of values connecting the area of the shaded portion with the length of AP, if P moves along AD; represent this relation graphically and find the length of AP for which the shaded area is least.

9. A stone is projected along a smooth horizontal surface in a resisting medium; it is found that it travels $15t - \frac{1}{5}t^3$ feet in t seconds. Represent graphically the function $15t - \frac{1}{5}t^3$ for values of t from 0 to 8. How far does the stone go before it stops?

10. If a man is h feet above sea-level, the horizon is about $\sqrt{\left(\frac{3h}{2}\right)}$ miles away. Make a table of values by taking for h the values 0, 6, 24, 54, 96, 121·5, 150, 181·5, 216. [Why are these values for h suggested?].

Represent this function of h graphically. Use the graph to find how far off the Skerryvoore light (139 ft. above sea-level) is visible from a point at sea-level. At what height is the horizon 10 miles distant?

11. The area of a rectangle is 48 sq. inches. If one side is x in. long, what is the length of an adjacent side? Show by a graph how the length of the adjacent side alters as x increases from 2 to 16. By using your ruler, find from the graph the length of one side if it is one-quarter of the length of the adjacent side.

12. If a given mass of gas occupies v cu. in. when under a pressure of p lb. per sq. in., then (by Boyle's Law), $p \,.\, v$ is constant if the gas is allowed to expand or is compressed, the temperature being constant. For a certain mass of gas, the relation is $p \,.\, v = 5000$. Represent this graphically as p varies from 50 to 100.

13. If a horse is pulling a vehicle at v miles an hour along a level road it exerts a pull of P lb.-wt. where $P \,.\, v = 330$. Represent graphically the relation between P and v. [The horse-power at which this horse is working is $P \cdot \frac{44v}{30} \times \frac{1}{550}$. How much is this ?]

14. From a sheet of cardboard, equal squares, side a in., are cut away at each corner (see Fig. 163). A box is then formed

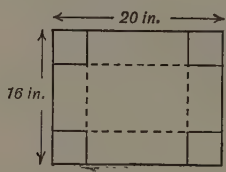


FIG. 163.

by folding along the dotted lines. Express, in terms of a, the volume of the box and draw the graph of this function of a as a increases from 0 to 8. For what value of a, is the volume of the box greatest ?

15. When a rod 10 in. long, pivoted at one end, swings like a pendulum, the tendency to break at a point h in. below the pivot is measured by the expression $h(10-h)^2$. Represent this function graphically for values of h from 0 to 10 and find where the rod is most likely to break.

16. The legs of a compass are each 5 cm. long. Find by measurement the distances between their points when the angles between the legs are 40°, 60°, 80°, 120°, 140°, 160°. Represent graphically the relation between the distance and the angle. [The graph is called a sine-curve.]

Can you use the graph to solve the following : the string of a kite is 100 yards long and makes an angle of 67° with the ground ; what is the height of the kite ?

Revision.

By examining the series of graphs which have been drawn in the course of this chapter and in Chapter V., the reader should now be able to appreciate and summarise the general principles underlying graphical work.

(i) The significance of a series of facts or statistics can often be realised more easily if the facts are presented in a pictorial form rather than as groups of printed figures.

(ii) A formula represents the relation between two *or more* numbers or quantities, *e.g.* $A=\pi r^2$; $V=\pi r^2 h$; $H=\dfrac{Wh}{33000t}$.

A graph represents the relation between two numbers or quantities, one of which is treated as the independent variable, whose value determines the value of the other, called the dependent variable.

(iii) Irregular variation produces an irregular line for the graph.

(iv) A graph that represents the variation of two quantities according to an observed law of Nature or a mathematical law embodied in a formula is, in general, a *smooth* line or curve.

(v) The graph is a *straight line*

(*a*) when one quantity is directly proportional to the other; or

(*b*) when one quantity rises or falls by equal steps when the other does so; or

(*c*) when the function represented is of the first degree.

(vi) The graph is a smooth curve

(*a*) when one quantity varies inversely as the other or varies as any power of the other; or

(*b*) when one quantity rises or falls by varying amounts when the other is increasing by constant amounts; or

(*c*) when the function represented is not of the first degree.

(vii) The slope of the graph represents the rate of increase (or decrease) of the function compared with that of the independent variable.

CHAPTER VII

DIRECTED NUMBERS

THE plan of this chapter is as follows :

(i) The *practical* advantage of introducing a new form of number—the directed number—is pointed out and the meaning of such numbers is explained.

(ii) Illustrations are then taken to show what is *meant* by addition or subtraction, applied to directed numbers, and this leads to the formulation of the necessary "rule of signs" for addition and subtraction.

(iii) Multiplication and Division are treated in the same way.

(iv) Exercises are included to secure facility in working with positive and negative numbers and in dealing with brackets.

(v) There is also a short section showing how directed numbers sometimes appear in the solution of problems.

Sections (i)-(iii) are intended to provide the basis for an oral treatment, *which can be easily abbreviated if circumstances make this desirable.*

Signless Numbers.

There are 5 people in this room ; you can arrive at this fact by pointing and counting 1, 2, 3, 4, 5. The 5 here used to describe the number of people is a signless number. There is no meaning in saying there are (-5) people in the room and no meaning in saying there are $(+5)$ people in the room, for $(+5)$ is merely a contrast to (-5).

We can add any two signless numbers or quantities together, 10 cows + 4 cows = 14 cows, and we can subtract, provided that what we subtract is not greater than the number or quantity from which we subtract it, 10 cows − 4 cows = 6 cows, but 10 cows − 12 cows is nonsense. *Here the signs + and − are commands to perform an operation, so also are* × *and* ÷, *provided that the operation is possible.*

Thus 10 cows × 3 = 30 cows and 10 cows ÷ 2 = 5 cows, and 10 pints ÷ 3 = $3\frac{1}{3}$ pints.

In quantities of this kind, there is no idea of "up and down" or "backwards and forwards" or "clockwise and anti-clockwise," and therefore signless numbers supply all that is needed for their expression and measurement.

Positive and Negative Numbers.

It often happens, however, that we wish to express in *short-hand form* facts for which the use of signless numbers is unsuitable or inadequate.

Suppose I buy a number of things and sell them again, the result of the transactions may be recorded as follows:

	House.	Car.	Picture.	Horse.	Field.	Carpet.	Chest.
Gain	£80			£25	£60		£8
Loss		£40	£15			£12	

In this table, the numbers are signless: but by inventing what are called **directed numbers**, we can express the results more shortly, as follows:

	House.	Car.	Picture.	Horse.	Field.	Carpet.	Chest.
Gain	£(+80)	£(–40)	£(–15)	£(+25)	£(+60)	£(–12)	£(+8)

The symbols + and – in this table are *not* instructions to add or subtract, they are called the **signs** of the numbers; the number (+80) is called a **positive number**, the number (–40) is called a **negative number**. They form a new kind of *short-hand notation*: thus my gain of £(+80) means that I am £80 better off by the transaction and my gain of £(–40) means that I am £40 worse off by the transaction. In one case my capital has *gone up* by £80, in the other case it has come down by £40. This new notation enables us to express in a short-hand form up-and-down movements, or similarly backwards-and-forwards movements, *i.e.* movements with which the idea of direction is associated. We therefore call numbers of this kind **directed numbers**.

Exercise VII. *a*. (Oral.)

1. A tank, with its base horizontal, contains water to a depth of 10 in., a number of vertical rods A, B, C, ... are fixed to the base (see Fig. 164).

	A	B	C	D	E	F
Length of rod in inches -	18	8	4	12	7	15

Express in short-hand form, the *height* of the upper end of each rod *above* the water level.

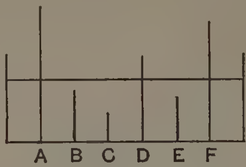


FIG. 164.

2. On a Centigrade thermometer (see Fig. 165), the freezing point of water is indicated by 0° C.; express in short-hand form, the following temperatures :

(i) 4° below freezing point ;
(ii) 20° above freezing point ;
(iii) 15·5° above freezing point ;
(iv) 22·8° below freezing point ;
(v) $7\frac{1}{2}$ degrees of frost.

FIG. 165.

What is the meaning of (i) (-5) degrees Centigrade ; (ii) $(+18)$ degrees Centigrade ; (iii) a rise in temperature of $(-3°)$; (iv) a fall in temperature of $(-4°)$?

3. A number of clocks are being regulated ; express in short-hand form the following records :

Clock.	I	II	III	IV	V	VI	VII
Seconds fast	17		8			3	5
Seconds slow		8		11	14		

4. A gun is being ranged by an aeroplane on a target ; the direction is correct ; the distance of the fall of the shell from the target is signalled as follows :

Round.	I	II	III	IV	V	VI
Distance in yards -	+260	−180	+140	−70	+35	O.K.

What do these signals mean ?

5. Explain the following :
Height in feet above sea-level.

Winchester.	Dead Sea.	Jerusalem.	Hill 60, Ypres.	Sea of Galilee.
$(+128)$	(-1300)	$(+2500)$	$(+195)$	(-680)

6. Express in short-hand form :

To travel from *O* to - -	*A*	*B*	*C*	*D*	*E*	*F*	*G*
Proceed miles east - -	24				5	7	
Proceed miles west - -		24	8	10			17

7. Taking the year 1914 A.D. as the beginning of a new era 0 A.B., express the following dates by directed numbers 1913 A.D., 1915 A.D., 1900 A.D., 1926 A.D., the year of your birth.

8. Express in short-hand form a man's banking account.

	Jan. 1.	March 1.	May 1.	July 1.	Sept. 1.	Nov. 1.
Credit -		£70	£30			£12
Overdraft -	£55			£43	£49	

9. What do the following golf handicaps mean :

(i) $+4$; (ii) -12 ; (iii) -18 ; (iv) $+1$; (v) 0 ?

10. A stone is thrown vertically upwards with velocity 40 ft. per sec. Explain the meaning of the following table which shows its velocity at half-second intervals :

Time in sec. - -	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
Velocity in ft. per sec.	$+40$	$+24$	$+8$	-8	-24	-40	-56

11. Rewrite the following, using signless numbers :

(i) A temperature of (-8) degrees C.

(ii) The lowest part of the bed of Lake Como is (-600) ft. above sea-level.

(iii) My watch is (-3) minutes fast, and yours is (-2) minutes slow.

(iv) My bank balance is £(-24).

(v) In a 220 yards race, I received a start of (-10) yards.

(vi) This year, I have gained (-10) lb. in weight and you have lost (-15) lb. in weight.

12. Taking Greenwich time as the standard, the following variations occur in local time :

	Berne.	Halifax.	Petrograd.	New York.
Hours	$+1$	-4	$+2$	-5

What does this mean ?

13. Taking 12 noon as zero hour, express by directed numbers the following times, (i) 3 p.m., (ii) 11 a.m., (iii) 8 a.m., (iv) 4.30 p.m., (v) 10.30 a.m.

14. Write down four consecutive whole numbers of which the greatest is (i) $+10$, (ii) $+2$, (iii) -2.

15. A is 2 miles north of B; how many miles is B north of A ?

The Number-Scale.

We may represent positive and negative numbers on a scale (see Fig. 166).

Such a scale is frequently of practical value, as for example in the case of the scale for the Centigrade thermometer, where temperatures above the freezing point of water are represented by positive numbers and those below freezing point by negative numbers.

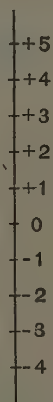


FIG. 166.

Addition and Subtraction.

Example for Oral Discussion. If the temperature (using the Centigrade scale) is $(+1°)$ at 3 p.m. and then falls $(+3°)$ in the next hour, it is $(-2°)$ at 4 p.m.

From $(+1°)$ we subtract $(+3°)$ and obtain $(-2°)$.

In symbols, $(+1°)-(+3°)=(-2°)$.

Now a fall of temperature $(+3°)$ can be written as a rise of temperature $(-3°)$.

We can therefore say that the temperature is $(+1°)$ at 3 p.m. and rises $(-3°)$ in the next hour.

$\therefore$ the temperature at 4 p.m. is $(+1°)+(-3°)$; but this is $(-2°)$.

To $(+1°)$ we add $(-3°)$ and obtain $(-2°)$.

In symbols, $(+1°)+(-3°)=(-2°)$.

Therefore the effect of *subtracting the positive number* $(+3)$ is exactly the same as the effect of *adding the negative number* (-3) and each is obtained by moving 3 steps down the scale (Fig. 166).

Using symbols, we say that $+(-N)$ *can be replaced by* $-(+N)$, *which means "subtract the number* $(+N)$*" or "move N steps down the scale."*

If the temperature is $(+1°)$ at 3 p.m. and then rises $(+3°)$ in the next hour, it is $(+4°)$ at 4 p.m.

In symbols, $\quad (+1°)+(+3°)=(+4°)$.

But a rise of $(+3°)$ can be written as a fall of $(-3°)$. We can therefore say that the temperature is $(+1°)$ at 3 p.m., and then falls $(-3°)$ in the next hour.

$\therefore$ the temperature at 4 p.m. is $(+1°)-(-3°)$; but this is $(+4°)$.

In symbols, $\quad (+1°)-(-3°)=(+4°)$.

Therefore the effect of *adding the positive number* $(+3)$ is exactly the same as the effect of *subtracting the negative number* (-3), and each is obtained by moving 3 steps up the scale (Fig. 166).

Using symbols, we say that $-(-N)$ *can be replaced by* $+(+N)$, *which means "add the number* $(+N)$*" or "move N steps up the scale."*

We therefore have the following **Rule of Signs**:

$+(-N)$ can be replaced by $-(+N)$: move N steps down the scale.

$-(-N)$ can be replaced by $+(+N)$: move N steps up the scale.

Note. The brackets are inserted to avoid ambiguity, *i.e.* to distinguish $+2$ meaning "add 2" from $(+2)$ meaning "the number plus 2." Again, $-(-N)$ means "subtract the number $(-N)$" and $+(+N)$ means "add the number $(+N)$"; each of these operations produces the same effect.

Exercise VII. *b*. (Oral.)

1. The temperature is $(+5°)$ C. If it rises $(+7°)$, what does it become? If it falls $(+7°)$, what does it become?

What is (i) $(+5°)+(+7°)$, (ii) $(+5°)-(+7°)$?

2. The temperature is $-3°$ C. If it rises $(+5°)$, what does it become? If it falls $(+5°)$, what does it become?

What is (i) $(-3°)+(+5°)$, (ii) $(-3°)-(+5°)$?

3. The temperature is $(+5°)$ C. It then rises $(-4°)$, what does this mean and what does the temperature become?

What is $(+5°)+(-4°)$?

4. The temperature is $(+5°)$ C. It then falls $(-4°)$, what does this mean and what does the temperature become?

What is $(+5°)-(-4°)$?

5. The temperature rises from $(+5°)$ C. to $(+7°)$ C.; what is the rise? The temperature rises from $(-5°)$ C. to $(+7°)$ C.; what is he rise?

What is (i) $(+7°)-(+5°)$, (ii) $(+7°)-(-5°)$?

6. Fill in all the blank spaces in the following table:

Temp. at 11 a.m. Jan. 1	$(+7°)$	$(+6°)$	$(-5°)$	$(+7°)$	$(-7°)$	$(-7°)$
Temp. at 10 a.m. Jan. 1	$(+3°)$	$(-2°)$	$(-9°)$	$(+10°)$	$(+10°)$	$(-3°)$
Rise	—	—	—	—	—	—
Fall	—	—	—	—	—	—

7. What must be added to

(i) $(+3)$ to give $(+7)$; (ii) $(+3)$ to give (-2);
(iii) (-5) to give (-2); (iv) (-5) to give $(+4)$;
(v) (-5) to give (-8); (vi) (-5) to give 0;
(vii) $(+4)$ to give 0; (viii) 0 to give (-3)?

8. Write down the values of:

(i) $(+8)+(-4)$; (ii) $(-7)+(-3)$; (iii) $(+2)-(+4)$;
(iv) $(-2)-(-3)$; (v) $(-6)-(-5)$; (vi) $(-9)+(+9)$;
(vii) $(-2)-(+3)$; (viii) $(-3)-(-2)$; (ix) $(-2)-(+2)$;
(x) $0-(+2)$; (xi) $0-(-2)$; (xii) $(+2)-(-3)$;
(xiii) $0+(-4)$; (xiv) $(-6)-(-6)$; (xv) $(-5)+(-5)$;
(xvi) $(-3)-0$; (xvii) $(+4)+0$; (xviii) $(-7)-(-10)$.

9. A is $(+200)$ feet above sea-level: what is the height of B above sea-level, if B is (i) 50 ft. above A; (ii) 100 ft. below A; (iii) 350 ft. below A?

10. Working in feet above sea-level, the surface of Lake Como is $(+700)$ and the lowest point of its bed is (-600). What is the greatest depth of the lake?

11. On a map of the west coast of Scotland, the heights of three "bench-marks" are given as 14 ft., 50 ft., 83 ft. above sea-level. What were their heights when the whole country was 50 ft. lower than it is at present?

12. Express by an equation: a gain of £5 + a loss of £3 is equal to a gain of £10 + a loss of £8.

13. Express a gain of £5 + a loss of £8 as a gain.

14. A man weighs himself each year on his birthday.

1920.	1921.	1922.	1923.	1924.	1925.
170 lb.	175 lb.	162 lb.	156 lb.	160 lb.	168 lb.

Express the annual changes as increases.

15. A man starts the year with a net deficit of £240 and ends the year with a deficit of £210. Can you use this to illustrate the value of $(-210)-(-240)$?

16. A miner starts from a place 140 ft. below sea-level and ascends to the top of a hill 80 ft. above sea-level. Use this to illustrate the value of $(+80)-(-140)$.

17. A freezing mixture is at $(-16°)$ C.; it is melted and heated up to $(+12°)$ C. How many degrees has its temperature risen?

18. Which is the greater (-3) or (-6)? Illustrate (i) from the number scale, (ii) from the temperature scale, (iii) from the idea of the value of an estate after paying debts due on it, (iv) from the idea of sea-level.

19. A commercial traveller receives a fixed salary of £200 a year; he also receives commissions which vary from year to

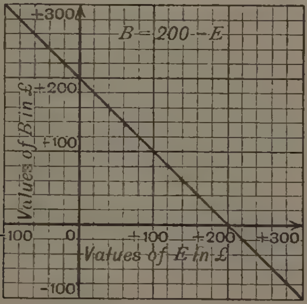


FIG. 167.

year. If his *net* annual expenditure (that is to say what he spends less what he receives in commissions) is £E, his balance on the year, £B, is given by the formula $B=200-E$. The graph in Fig. 167 represents the relation between B and E.

Use the graph to answer the following questions and write down the corresponding numerical calculation.

What is his balance on the year if his net expenditure is (i) £100, (ii) £160, (iii) £200, (iv) £260 ?

What is his balance on the year, if $E = (-40)$ and what does this mean ?

What is his net expenditure if his balance on the year is (i) £140, (ii) £300, (iii) £(-100) ?

Write down the values of the following :

20. $(+6x)+(-x)$.
21. $(-2a)+(+a)$.
22. $(-3b)+(-4b)$.
23. $(-y)+(+y)$.
24. $(+3c)+(-10c)$.
25. $(-3a)+(-8a)$.
26. $(-3z)-(+2z)$.
27. $(+2x)-(-4x)$.
28. $(+c)-(+3c)$.
29. $(-6x)-(-8x)$.
30. $(+5a)-(-5a)$.
31. $0-(+6b)$.
32. $(-3a^2)+(+4a^2)$.
33. $(+7ab)+(-2ab)$.
34. $(+10y)-(-2y)$.
35. $(-x^2)-(+x^2)$.
36. $(-a^2)+(-5a^2)$.
37. $0+(-3c)$.
38. $(+x)+(+2x)+(-4x)$.
39. $(+3a)+(-5a)-(+2a)$.
40. Write down (i) the term of degree 2, (ii) the coefficient of x, (iii) the constant term in

(a) $3x^2-7x-2$; (b) $2x^3-5x^2-x+4$; (c) x^4-x^2+3x-1.

Multiplication and Division.

It is meaningless to talk of the sum or difference of two different *kinds* of quantities, *e.g.* 2 days + 5 inches or £2 − 5 pints ; in the same way, it is meaningless to talk of the sum or difference of a *signless* number and a *directed* number. If an expression is written $2-(-3)$ or $(-5)+1$, it is really an abbreviation of $(+2)-(-3)$, $(-5)+(+1)$.

There is no difficulty in multiplying or dividing a directed number by a signless number.

Thus $(-5)\times 3=(-5)+(-5)+(-5)=(-15)$.

And this shows that $(-15)\div 3=(-5)$.

Oral Example. A builder sells five houses and loses £100 on each house. What is (i) his *gain* on each house, (ii) his total *gain* ?

Product of Two Directed Numbers.

Example for Oral Discussion. Suppose that the temperature of the water in a boiler is being raised at a steady rate of 5° C. per hour throughout the day and that by mid-day it has become 40° C.

Let us call mid-day zero hour; then 2 p.m. is $(+2)$ hours after mid-day or $(+2)$ o'clock, and 9 a.m. is (-3) hours after mid-day or (-3) o'clock; and now let us find how much the temperature at (say) n o'clock is above that at zero hour (mid-day).

The rise per hour is $(+5°)$.

$\therefore$ the temperature at n o'clock is $(+5°) \times n$ above the temperature at zero hour.

Here, *n is a directed number*; for, at 2 p.m., $n=(+2)$, and at 9 a.m., $n=(-3)$.

$\therefore$ at 2 p.m., the temperature is $(+5°) \times (+2)$ above that at noon; and at 9 a.m., the temperature is $(+5°) \times (-3)$ above that at noon.

But at 2 p.m. the temperature is ten degrees above that at noon, so that the amount above is $(+10°)$.

And at 9 a.m., the temperature is fifteen degrees below that at noon, so that the amount above is $(-15°)$.

$$\therefore \ (+5) \times (+2) = (+10) \quad \textbf{and} \quad (+5) \times (-3) = (-15).$$

Next suppose that the temperature of the water in the boiler is *falling* at a steady rate of 5° C. per hour throughout the day. Take mid-day as zero hour and find how much the temperature at n o'clock is above that at zero hour (mid-day).

The rise per hour is $(-5°)$.

$\therefore$ the temperature at n o'clock is $(-5°) \times n$ above the temperature at zero hour.

$\therefore$ at 2 p.m., the temperature is $(-5°) \times (+2)$ above that at noon; and at 9 a.m., the temperature is $(-5°) \times (-3)$ above that at noon.

But at 2 p.m., the temperature is ten degrees below that at noon, so that the amount above is $(-10°)$.

And at 9 a.m., the temperature is fifteen degrees above that at noon, so that the amount above is $(+15°)$.

$$\therefore \ (-5) \times (+2) = (-10) \quad \textbf{and} \quad (-5) \times (-3) = (+15).$$

We can apply this argument to any directed numbers: we are therefore led to the following **rule of signs**:

$$(+a) \times (+b) = (+ab) = (-a) \times (-b),$$
$$(+a) \times (-b) = (-ab) = (-a) \times (+b).$$

Division.

Since $(+5) \times (+2) = (+10)$, it follows that $(+10) \div (+2) = (+5)$,

Similarly, since $(+5) \times (-2) = (-10)$; $\therefore (-10) \div (-2) = (+5)$,

since $(-5) \times (+2) = (-10)$, $\therefore (-10) \div (+2) = (-5)$,

since $(-5) \times (-2) = (+10)$, $\therefore (+10) \div (-2) = (-5)$.

We therefore have the following **rule of signs :**

$$(+\mathrm{a})\div(+\mathrm{b})=\left(+\frac{\mathrm{a}}{\mathrm{b}}\right)=(-\mathrm{a})\div(-\mathrm{b}),$$

$$(+\mathrm{a})\div(-\mathrm{b})=\left(-\frac{\mathrm{a}}{\mathrm{b}}\right)=(-\mathrm{a})\div(+\mathrm{b}),$$

provided that b is not zero.

Note. (i) If any number is multiplied by 0, the product is 0. We shall never speak of dividing a number by 0.

(ii) Since $(+a)\times(+a)=(+a^2)=(-a)\times(-a)$, we see that both $(+a)$ and $(-a)$ are square roots of $(+a^2)$.

A positive directed number has therefore two square roots; but the symbol $\sqrt{(+a^2)}$ or $\sqrt{(a^2)}$ is used to mean the *positive* square root.

The Rules of Sign for multiplication and division given above may be expressed as follows :

In multiplication and division of one directed number by another, like signs give a positive sign and unlike signs give a negative sign.

Oral Example I. A retired tradesman is living partly on his savings. He finds that his bank balance is being diminished at the rate of £60 per annum. At present he has £300 in the bank. If £B represents his balance at the end of n years, the formula for finding B is

$$B=(+300)+(-60)\times n.$$

In Fig. 168, PQ is the graph showing the relation between the years to come and his bank balance from now until $(+5)$.

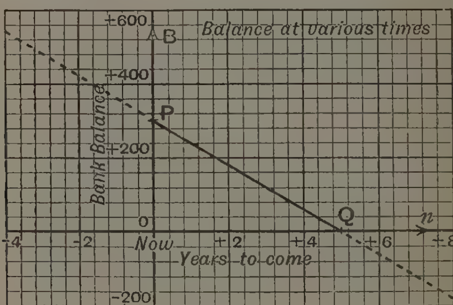


FIG. 168.

If he has been living in the same way for some years previously and continues to go on in the same way beyond 5 years, the dotted extensions are part of the graph.

Either use the printed graph, or preferably reproduce the graph on a larger scale and use your own figure, to answer the following:

(i) What was his balance 2 years ago, *i.e.* when $n = -2$?

(ii) What will be his balance in 6 years to come?

(iii) What is B when $n = +3$, $n = -3$, $n = +7$, $n = -1\frac{1}{2}$?

(iv) What is n when $B = +60$, $B = +420$?

Afterwards, use the formula and the rule of signs to calculate the answers to these questions.

Exercise VII. *c*.

1. A man pays into a savings-bank £10 on the last day of each month. On June 1st, he has £90 in the bank. Take this as zero date and measure the time in months. How would you represent Sept. 1, April 1 (same year)?

Represent by directed numbers, without simplifying, his banking account, (i) on Sept. 1, (ii) on April 1, assuming that he does not draw out any money from the bank.

How much is $(+10) \times (+3)$ and $(+10) \times (-2)$?

2. Repeat No. 1, supposing that the man draws £10 out of the bank on the last day of each month, instead of paying it in?

How much is $(-10) \times (+3)$ and $(-10) \times (-2)$?

3. A motor-car, travelling at half a mile a minute due North from Winchester to Oxford, passes through Newbury at midnight, zero hour. Draw a diagram to show the data and then represent by directed numbers how far the car is North of Newbury at 6 minutes past zero hour and at 8 minutes before zero hour. Then simplify the expressions.

4. Suppose in No. 3, the car is travelling at half a mile a minute due South; how can you represent its velocity northwards? Draw a diagram to show the data and then represent by directed numbers how far the car is North of Newbury at 10 minutes past zero hour and at 4 minutes before zero hour. Then simplify the expressions.

5. The population of a village decreases by 12 every year. At present it is 850; show that in x years' time it will be $(+850) + (-12) \times (+x)$. What is the meaning and result if for $(+x)$ we write (i) $+5$, (ii) -10?

6. A farmer says that he loses £100 every year. At the present moment his capital is £4500.

Find a formula for his capital in n years' time. What is the meaning and result if for n we write (i) $+3$, (ii) -4?

7. Repeat No. 6, supposing the farmer is really gaining £100 every year.

8. A car is travelling Eastwards at v feet per sec. (see Fig. 169); at a time t seconds after passing A it is s feet East of A; show that $s = vt$.

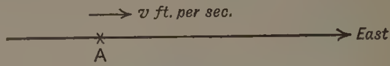


FIG. 169.

Evaluate s and interpret the data and results, if (i) $v = +50$, $t = +10$; (ii) $v = +50$, $t = -10$; (iii) $v = -50$, $t = +10$; (iv) $v = -50$, $t = -10$.

9. A man looking out of a top-storey window A sees a stone pass the window moving vertically upwards (see Fig. 170); t seconds after passing A, the stone is s ft. above A, where $s = t(+24) + t^2(-16)$. Evaluate s and interpret the data and results if t equals (i) $-\frac{1}{2}$; (ii) 0; (iii) $+\frac{1}{2}$; (iv) $+1$; (v) $+\frac{3}{2}$; (vi) $+2$.

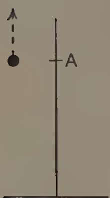


FIG. 170.

10. If we take the numbers $+18$, $+14$, $+10$, $+6$, ... and add up the first n of them, the sum is $(+20) \times n + (-2) \times n^2$. Verify this for $n = 1$, $n = 2$, $n = 3$. Evaluate the expression when $n = 5$, 6, 7, 10, and interpret the result.

EXERCISE VII. *d*.

Write down the values of :

1. $(-10) \times 3$.
2. $(-1) \times 8$.
3. $(-5) \times (-6)$.
4. $(-2) \times (-8)$.
5. $(-1) \times (-3)$.
6. $(+10) \div (-5)$.
7. $(-4) \div (-1)$.
8. $(+48) \div (-12)$.
9. $(-36) \div (-9)$.
10. $(-18) \div 6$.
11. $(-10) \div (-10)$.
12. $(-3)^2$.
13. $(-1)^2$.
14. $(-2)^3$.
15. $(-3)(-6) \div (-9)$.
16. $(-6)^2 \div (-4)$.
17. $(-8)(+3) \div (-6)$.
18. $(+36) \div (-3)^2$.
19. $0 \times (-3)$.
20. $(-2) \times 2$.
21. $0 \div (-4)$.
22. $\dfrac{(-2)}{(-4)}$.
23. $0 \div (+1)$.
24. $\dfrac{(+6)}{(-3)}$.
25. $(+2a)(-3a)$.
26. $(-3a)(+2a)$.
27. $(-x)(+y)$.
28. $(-6a^2) \div (+2a)$.
29. $(+10xy) \div (-2y)$.
30. $(+14x^2) \div (-7x)$.
31. $(+3x)(-5)$.
32. $(-3x^2)(+2)$.
33. $(+ab)(-1)$.

34. $(-20xy)\div(-5)$.
35. $(+4x^2)\div(-1)$.
36. $(-9a^2)\div(-9a)$.
37. $(-2a)(-3b)$.
38. $(6a^2)(-4a)$.
39. $(+1)(-3xyz)$.
40. $(+3xyz)\div(-xyz)$.
41. $(-12x^4)\div(-x)$.
42. $(+4a^2b^2)\div(-2ab)$.
43. $(-2a)(-2bc)$.
44. $(-x^2y)(-xy^2)$.
45. $0\times(-2a)$.
46. $(+15x^3y^3)\div(-5x^2y)$.
47. $0\div(-x)$.
48. $(+abc)\div(-abc)$.
49. $(-y)(+y)$.
50. $(-xy)\div(-xy)$.

If $a=+3$, $b=-3$, $c=0$, $m=+2$, $n=-2$, $x=+1$, $y=-1$, $z=-2$, find the values of :

51. $a+x$.
52. $b+x$.
53. $b-x$.
54. $a+b$.
55. $c+x$.
56. $a-2c$.
57. $2x-b$.
58. $2(x-b)$.
59. $b-2x$.
60. $n-3m$.
61. $3n-2y$.
62. ab.
63. nx.
64. $2an$.
65. $3n^2$.
66. abc.
67. a^2-ab.
68. b^2-b.
69. $2(y+z)$.
70. $x-2(y-z)$.
71. x^2+y^2.
72. $\frac{x}{y}$.
73. $(x-y)(y-z)$.
74. $(x-y)^2$
75. $\frac{1}{x}+\frac{1}{y}$.
76. $\frac{a}{b}+\frac{m}{n}$.
77. $(a+b)^2+z^2$.
78. $\frac{x-2y}{z}$.

Brackets.

We have hitherto enclosed positive and negative numbers in brackets, whenever there is any possible ambiguity ; but in future we shall usually omit the brackets, and further, for $(+1)$, $(+2)$, $(+3)$, etc. we shall write simply 1, 2, 3, etc.

Brackets which bind two or more directed numbers together may be removed by precisely the same rules as apply to signless numbers (see p. 64).

Example II. Simplify

(i) $+(-2a)$; (ii) $-3(+2b)$; (iii) $(-2)(-3c)$; (iv) $-(+3d)$.

(i) $+(-2a)$ means "add the number $(-2a)$," which is the same as "subtract $(+2a)$" and we write it $-2a$.

(ii) $-3(+2b)=-(+6b)=-6b$.

(iii) $(-2)(-3c)=(+6c)=6c$.

(iv) $-(+3d)=-3d$.

Example III. Simplify $(-8)-[(+5)-(+7)]$.

$$(-8)-[(+5)-(+7)]=(-8)-(+5)+(+7)$$
$$=-8-5+7=-13+7=-6.$$

Example IV. Simplify $(+3a)-4[(-2b)-(+3c)]$.

$$\begin{aligned}\text{The expression} &= (+3a)-4(-2b)+4(+3c)\\ &= +3a-(-8b)+(+12c)\\ &= 3a+8b+12c.\end{aligned}$$

Exercise VII. *e*.

Simplify the following and remove the brackets :

1. $-(-2x)$. 2. $+3(+y)$. 3. $+(-2b)$. 4. $-3(-a)$.
5. $(+4)-(+2)+(+3)-(-5)$.
6. $-(+2)+(-3)-2(-1)+(+5)$.
7. $(-2a)-(-2a)-(+2a)+(+3a)$.
8. $-(-4p)+(-3q)-(+2r)$.
9. $(-5)+2[(+3)-(-2)]$. 10. $3(-1)-[(-2)+(+4)]$.
11. $(-1)(+1)-[(-1)-(+1)]$. 12. $2[(-3)+(+1)]-3(-2)$.
13. $(-2b)-3[(+b)-(-2b)]$.
14. $3[(-x)-2(-y)]-[(+x)-(-2y)]$.
15. Express in bracket form in two ways :
   (i) 6 ; (ii) -3 ; (iii) $-2p$; (iv) $4z$.
16. Fill in the blanks in the following :
   (i) $2a-6b=(+2)[\qquad]=(-2)[\qquad\qquad]$;
   (ii) $-p+3q=(-1)[\qquad]=+[\qquad]$.

Directed Numbers in Problems and Equations.

Example V. A slow train S travelling 30 miles an hour, and a fast train F travelling 45 miles an hour are both proceeding due East.

Eight minutes after S has passed a level crossing L, S is 3 miles behind F. Find the distance from L of the place P at which F passed S.

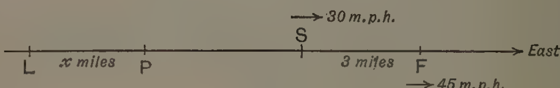


Fig. 171.

Let P be x miles east of L.

S travels 30 miles in 1 hour or $\frac{1}{2}$ mile in 1 minute or 4 miles in 8 minutes.

$$\therefore\ LS=4 \text{ miles};\quad \therefore\ PS=(4-x) \text{ miles};$$
$$\therefore\ PF=4-x+3=(7-x) \text{ miles}.$$

S travels 1 mile in 2 minutes, F travels 1 mile in $\frac{60}{45} = \frac{4}{3}$ minutes.

$\therefore$ S takes $2(4 - x)$ minutes for the distance PS, and F takes $\frac{4}{3}(7 - x)$ minutes for the distance PF.

But these times are the same, since they passed at P.

$$\therefore\ 2(4 - x) = \tfrac{4}{3}(7 - x);$$
$$\therefore\ 6(4 - x) = 4(7 - x);$$
$$\therefore\ 24 - 6x = 28 - 4x \quad \text{or} \quad 4x - 6x = 28 - 24;$$
$$\therefore\ -2x = 4, \text{ multiply each side by } -1.$$
$$\therefore\ 2x = -4 \quad \text{or} \quad x = -2.$$
$$\therefore\ P \text{ is } (-2) \text{ miles east of } L.$$

This means that F passes S at a place 2 *miles west* of L.

Example VI. Solve the equation $1 - 2\frac{1}{2}p = 7$.

$$1 - 2\tfrac{1}{2}p = 7, \quad \therefore\ -2\tfrac{1}{2}p = 7 - 1 = 6;$$
$$\therefore\ -5p = 12, \text{ multiply each side by } -1;$$
$$\therefore\ 5p = -12;$$
$$\therefore\ p = -\tfrac{12}{5} = -2{\cdot}4.$$

Exercise VII. *f*.

1. The present ages of A and B are 21 and 35; in n years' time, B will be twice as old as A. Find n and interpret the answer.

2. Two large kettles are being heated, A on a gas stove and B on a primus; at t minutes past eleven, the temperatures in degrees Centigrade of A and B are $30 + 2t$ and $48 + 5t$. At what time is the water at the same temperature in the two kettles? Each kettle was filled with water at 14° C.? At what times were the kettles put on the stoves?

3. Can you find six consecutive odd numbers whose sum is 12?

4. What number must be added to both numerator and denominator of the fraction $\frac{17}{25}$, so that the result is equal to $\frac{3}{5}$?

5. The heights of A and B above sea-level are a feet and b feet; and the height of A above C is equal to the height of C above B. Find the height of C. Interpret the answer when (i) $a = 100$, $b = -200$; (ii) $a = 50$, $b = -50$.

6. If $C°$ Centigrade is the same temperature as $F°$ Fahrenheit, $C = \frac{5}{9}(F - 32)$. Express 0° Fahrenheit in Centigrade. What temperature is represented by the same number on the two scales?

7. Two cars P, Q are travelling in the same direction along a road at u and v miles per hour respectively. At noon, Q is s miles ahead of P. At what time will P pass Q? What does your answer become if $u=20$, $v=24$, $s=2$, and what does it mean?

Solve the equations:

8. $1\frac{1}{2}p=-12$.

9. $-1\frac{3}{4}q=35$.

10. $7=-2t$.

11. $\dfrac{l}{l+1}=\dfrac{4}{3}$.

12. $\dfrac{x}{3}-\dfrac{x}{2}=1$.

13. $3(1-k)-5(2-k)=k-12$.

14. $\dfrac{1}{y}=-3$.

15. $A-\frac{5}{3}A=5$.

16. 15 years ago a father was three times the age of his son and 19 years ago he was four times the age of his son. How old are they now? In how many years' time will he be twice his son's age?

17. The marks obtained in an examination ran from 24 to 84; these were then scaled so as to run from 0 to 100. What was the scaled mark corresponding to n marks for the paper? A boy, who did the paper afterwards, obtained 15 marks for it, what would his mark become according to the same scale?

18. A small body P of weight W lb. is just displaced from the highest point A of a fixed sphere, whose diameter AB is d feet. In Fig. 172, AN represents the *vertical* distance P has fallen. When $AN=h$ feet, it can be proved that the pressure of the sphere on the body outwards from the centre O is $\dfrac{W}{d}(d-3h)$ lb.-wt. Evaluate this when P is level with O and interpret the result.

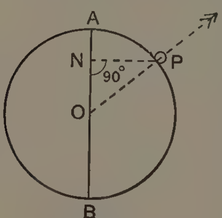


Fig. 172.

Find the depth of P below A when it leaves the surface of the sphere.

Example VII. Add $2c-a+b$ to $3a-b-c$.

$$\begin{array}{r} 3a-b-\ c \\ -a+b+2c \\ \hline 2a\qquad+\ c \end{array}$$

Example VIII. Subtract $2c-a+b$ from $3a-b-c$.

$$\begin{array}{r} 3a-\ b-\ c \\ -a+\ b+2c \\ \hline \mathbf{4a-2b-3c} \end{array}$$

Note. The terms can be subtracted as before, or you can change in your head the signs of the terms in the lower line and add; this depends on the following argument:

$$\begin{aligned}(3a-b-c)-(-a+b+2c) &= 3a-b-c+a-b-2c \\ &= (3a+a)+(-b-b)+(-c-2c).\end{aligned}$$

Example IX. Divide $3x^3-6ax^2+12a^2x$ by $-3x$.

$$\begin{array}{r|l} -3x & 3x^3-6ax^2+12a^2x \\ \hline & \mathbf{-x^2+2ax-4a^2} \end{array}$$

EXERCISE VII. *g*.

Simplify the following:

1. $2x-3x-x$.
2. $3x^2-5x-2x^2+x$.
3. $(2x-1)-(5x-1)$.
4. $a-2b+c+a+b-2c$.
5. $(a-b-c)(-3)$.
6. $(x^2-xy)-(xy-y^2)$.
7. $(-x^2y+xy^2)\div(-xy)$.
8. $(-x+y-z)+(-x-y+z)$.
9. $(-a^2+2a-1)(-2a)$.
10. $a(a-b)-b(a+b)$.
11. $x^2+2x-1-(2x^2+x-3)$.
12. $(z-y)-(y-z)$.
13. $(-2x^3-6x^2+10x)\div(-2x)$.
14. $(a-b)+(b-c)+(c-a)$.

Add:

15. $a-2b$ and $2a-b$.
16. $a-b+c$ and $a+b-c$.
17. $3x-2y$ and $-x+y$.
18. $-x$ and $2x-y$.
19. $2x-3$ and $x+1$.
20. $x-5$ and $-2x+1$.
21. $2x^2-x$ and $2x-1$.
22. $3a-2b$ and $b-2a$.
23. $x+y-z$ and $y+z-x$.
24. $1-x$ and $2+x-3x^2$.
25. $2-x$, $1-x^2$, $3-2x+x^2$.
26. $2(x^2-x-1)$ and $-3(x^2+2x)$.

Subtract:

27. $a+b$ from $2a-b$.
28. $x-y$ from $2x-3y$.
29. $2x-y$ from $x+y$.
30. $a-2b+c$ from $2a-b+c$.
31. $x-y-z$ from $x+y+z$.
32. $x-5$ from $2x-6$.
33. $1-x$ from $2x-1$.
34. $x-2$ from $3x$.
35. x^2-x+2 from $2x^2+x-1$.
36. $2x^2$ from x^2-1.
37. $a-2b+c$ from $2a-b+2c$.
38. $2x^2-x+1$ from x^2.
39. $2(x-1)$ from $-(2x-11)$.
40. $-(x+2)$ from $2x-5$.

Multiply:

41. $a+b$ by x.
42. $a+b$ by $-x$.
43. $a-b$ by $-x$.
44. $a-b$ by $-a$.
45. $x-1$ by $-x$.
46. $1-2a$ by $-b$.
47. $a-3x$ by $-2x$.
48. $a+b$ by $-ab$.
49. x^2-2x by -1.
50. $2x-3$ by $-5x$.
51. $4a+a^2$ by $-2a$.
52. $x-x^2$ by $-2x^2$.
53. $2x^2-x+3$ by $-x$.
54. $1-x+3x^2$ by $-2x$.

Divide:

55. $ab+ac$ by a.
56. $ab+ac$ by $-a$.
57. $ab-ac$ by $-a$.
58. x^2-2x by x.
59. x^3+x^2 by $-x$.
60. $4-4x$ by -4.
61. x^3-x by $-x$.
62. a^4-b^4 by -1.
63. x^2-x^6 by $-x^2$.
64. $2x^3-6x^2+8x$ by $2x$.
65. $3x^3-6x^2+18x$ by $-3x$.
66. $24+6x-18x^3$ by -6.
67. $5x^2y^2-10xy$ by $-5xy$.
68. $x^2-xy-xz$ by $-x$.
69. $a^3bc-abc^3$ by abc.
70. $-x^2yz+xy^2z+xyz^2$ by $-xyz$.

Simplify:

71. $a+(-2a)$.
72. $-x+(-x)$.
73. $2y-(-3y)$.
74. $xy-7xy$.
75. $-3x^3+(-x^3)$.
76. $-y-(-8y)$.
77. $(-6x)(5x)$.
78. $(-12x)\div(3x)$.
79. $8x^3\div(-2x)$.
80. $a^6\div(-a^3)$.
81. $(-2ab)(-ab)$.
82. $(-2a^3)\div(-a)$.
83. $x^2y^2\div(-y)$.
84. $-(2xy)^2$.
85. $(-2a)^3$.
86. $\dfrac{-x}{y}$.
87. $\dfrac{x}{-y}$.
88. $\dfrac{-x}{-y}$.
89. $\dfrac{-2ab}{b}$.
90. $\dfrac{-x}{-xy}$.
91. $\dfrac{a^2}{-ab}$.
92. $\dfrac{-a^2b}{-ab^2}$.
93. $\left(-\dfrac{a}{b}\right)\times b$.
94. $\left(-\dfrac{x}{y}\right)\times(-xy)$.
95. $\left(-\dfrac{1}{a}\right)\times\left(\dfrac{a}{2}\right)$.
96. $\left(\dfrac{x}{-1}\right)\left(\dfrac{-2}{x^2}\right)$.
97. Subtract $-x-y-z$ from 0.
98. Divide x^2-x+1 by -1.
99. Multiply x^2-2x^2 by $2x-3x$.
100. Simplify $\dfrac{x^2-xy}{-x}+\dfrac{y^2-xy}{-y}$.

EXTRA PRACTICE EXERCISES. E.P. 7.

Positive and Negative Numbers.

1. Subtract:
(i) $-a$ from a; (ii) b from $-b$; (iii) $5y$ from $7y$;
(iv) $-3x$ from $5x$; (v) $5p$ from $-2p$; (vi) $-6z$ from $-2z$.

2. Add:

(i) $-a$ and a; (ii) $-2b$ and $-3b$; (iii) $-c$ and $5c$;
(iv) $7x$ and $-4x$; (v) 0 and $-2y$; (vi) $-4z$ and $-z$.

3. Multiply:

(i) $-a$ by -1; (ii) $2b$ by -2; (iii) $-3x$ by -4;
(iv) $-y$ by y; (v) $-z$ by $-z$; (vi) -5 by $2c$.

4. Divide:

(i) $-4a$ by -1; (ii) -2 by -2; (iii) $6y$ by -2;
(iv) $-9x$ by 3; (v) $-x^2$ by $-x$; (vi) $6ab$ by $-3b$.

If $a=-1$, $b=-2$, $c=3$, $d=0$, $x=-4$, $y=1$, write down the values of the following, Nos. 5-57:

5. ab. 6. ac. 7. bd. 8. ax. 9. dy.
10. bc. 11. ay. 12. bx. 13. cd. 14. xy.
15. $\frac{b}{a}$. 16. $\frac{c}{a}$. 17. $\frac{x}{y}$. 18. $\frac{y}{a}$. 19. $\frac{a}{y}$.
20. $\frac{d}{y}$. 21. $\frac{d}{a}$. 22. $\frac{x}{b}$. 23. $\frac{b}{y}$. 24. $\frac{b}{x}$.
25. b^2. 26. a^3. 27. x^2. 28. a^4. 29. d^2.
30. $a-b$. 31. $b+c$. 32. $d-a$ 33. $x-y$. 34. $c-x$.
35. $y+a$. 36. $x-b$. 37. $a-c$. 38. $b-d$. 39. $y+b$.
40. $y-a$. 41. $x-c$. 42. $2x+y$. 43. $3b-c$. 44. $d-2a$.
45. $3y-b$. 46. $3b-2d$. 47. $b-2a$. 48. $a-2x$. 49. $c+4a$.
50. $2c-6a$. 51. $3y-c$. 52. $2b-x$. 53. $3x-4a$.
54. $a(b-c)$. 55. $b(d-a)$. 56. $x(y-b)$. 57. $c(a-x)$.

Simplify the following:

58. $(-p)(3p)$. 59. $(-2r)(-r)$. 60. $(+3s)(-2t)$.
61. $(-8t)\div(-2)$. 62. $(-6k)\div 3$. 63. $(4l)\div(-1)$.
64. $-(-4a)$. 65. $+(+3b)$. 66. $-(+2c)$.
67. $(-3)(2x)$. 68. $0\times(-3y)$. 69. $0\div(-4z)$.
70. $3p-5p-p$. 71. $a-4a+2a$. 72. $-t-3t-4t$.
73. $x+(-3x)$. 74. $(-y)+(-y)$. 75. $2z-(-z)$.
76. $-(-a)+(+a)$. 77. $-(+c)+(-c)$. 78. $+(-p)-(-p)$.
79. $-(+2b)-(-2c)$. 80. $3r-(-3s)$. 81. $(-2k)+(-l)$.
82. $(-a)\div(ab)$. 83. $c\div(-c^2)$. 84. $(-d)\div(-2d)$.
85. $\left(-\frac{p}{q}\right)\div(-p)$. 86. $\left(\frac{x}{y}\right)(-y)$. 87. $(-x^2)\div(xy)$.

88. $\left(-\frac{2}{r}\right)\times\left(\frac{r}{-2}\right)$. 89. $(-ab)\div(-2b)$. 90. $\left(\frac{a}{-1}\right)\times b$.

91. $(-1)(b-a)+(-a)$.
92. $c+(-2)(c-d)$.
93. $(x^2-2xy)\div(-x)$.
94. $(-3)(p-q)+(-2)(q-p)$.
95. $(r-s)\div(-1)+s$.
96. $-2y+(-3)(x-y)$.
97. $0+(-1)(z-y)$.
98. $(a-b)-(b-a)$.
99. $(-2)(3x)+(-1)(-4x)$.
100. $(-2x)^2-2x^2$.

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101. Subtract $-a-b+c$ from a.
102. Divide $x^2-3xy-x$ by $-x$.
103. Simplify $\frac{x}{-y}+\frac{-x}{y}-\frac{-x}{-y}$.
104. Fill in the blank space in $x-y=(-1)(\quad)$.
105. Simplify (i) $(-a)(-b)(-c)$; (ii) $(-x^3)^2$; (iii) $(-y)^2(-y^2)$; (iv) $(2p)(-3q)\div(-1)^2$.
106. Subtract $4x^2+7$ from x^2-5x-2.
107. Add $-(1-2x)$ to $3(x+1)$.
108. Divide $x^4y^2-x^2y^4$ by $-xy$.
109. Multiply $1-(2-a)$ by $-a$.
110. Simplify $\frac{b^2}{-b}+\frac{b^2-b^4}{-b^2}+(-b)^2$.
111. If $(x-a)(y-a)=c^2$, and if $a=-1$, $c=-1$, $y=1$, find the value of x.
112. If $r+s+4=0$ and $s=-1$, what is the value of $\frac{r}{s}$?
113. If $x=-\frac{1}{2}$ and $y=-\frac{1}{3}$, what is the value of $\frac{1}{x}-\frac{1}{y}$?
114. Simplify $(a-2a^2)\div(-a)+(2b^2-b)\div(-b)$.
115. If $pq=p+q$ and $q=-1$, what is p?
116. Can you find two numerical values of x such that the square of $x-7$ is 16?
117. Multiply $\frac{1}{-a}+\frac{1}{-b}-\frac{1}{-ab}$ by $-ab$.
118. Divide $x^3y^2z^2-x^2y^3z^2+x^2y^2z^3$ by $-xyz^2$.
119. Fill in the blank space in $2r-3s=(s-r)-(\quad)$.
120. If $y+z=0$, simplify $\frac{y^2+z^2}{yz}$.

MISCELLANEOUS EXAMPLES

M. III

1. What must be added (i) to -2 to give 5; (ii) to 2 to give -5; (iii) to $-a$ to give $-b$?

2. What is x, if (i) $-3x=12$; (ii) $\frac{x}{-3}=5$?

3. Add x, $-4x$, $5x$, $-3x$. Subtract $-x$ from the result.

4. A is 250 ft. above sea-level; B is (-50) ft. above sea-level. How much is A above B?

5. What must be subtracted from 0 to give -8?

6. What are a, b if $2x+5$ is the same as $a(x-b)$ for all values of

7. What are the values of (i) $\frac{3}{-1}$; (ii) $\frac{-2}{-1}$; (iii) $\frac{(-2)^3}{(-1)^3}$?

8. Multiply $-6x^3$ by $2x$ and divide the result by $-4x^2$.

9. What must be subtracted from $-a$ to give $-b$?

10. From x subtract $2x-y$.

11. If $y=3-7x$, what is y when x equals (i) 0; (ii) -1?

12. By how much does (i) x exceed $-y$; (ii) $-y$ exceed x?

13. Add together $-\frac{x}{5}$, $\frac{1}{-5}$, $\frac{x-4}{5}$.

14. On a slip, there are three rings for mooring boats; they are 2 ft., 5 ft., 9 ft. above the water. If the water rises 9 ft., what will be their heights above the water?

15. *Sketch roughly* the graph of $(-x)^2$ for values of x between 3 and -3. Sketch on the same figure the graph of $-x^2$.

16. Simplify $\frac{a-b+c}{b-a-c}$ and $\frac{a-(b+c)}{a-b-c}$.

17. The sum of two numbers is 0; one is $a-b$; what is the other?

18. What is x if (i) $\frac{x}{-2}=\frac{1}{4}$; (ii) $\frac{3}{-x}=\frac{1}{3}$?

19. Multiply $-\frac{1}{x}+\frac{1}{y}-\frac{1}{z}$ by xyz.

20. Fill in the blank space in $-\frac{1}{x}-\frac{1}{y}=-\frac{\quad}{xy}$.

21. There is a vertical ladder on the wall of a quay. The top is 25 ft., the bottom (-5) ft. above the water. The water rises till the top is 15 ft. above it: how much is the bottom now above it? What is the length of the ladder?

22. Simplify $(1-x)(-x)-(x-x^2)$.

23. Simplify $(x-2x^2)\div(-x)-(x-x^2)$.

24. Find the number which is halfway between (i) 5 and -3 (ii) 3 and -5; (iii) a and $-b$.

25. What does the statement $a^3+b^3 \equiv (a+b)(a^2-ab+b^2)$ become when you put $-x$ instead of b ?

26. What is the value of $(-1)^m$ when m is (i) an even integer; (ii) an odd integer ?

27. We say that x is greater than y if $x-y$ is positive. What does this give if (i) $x=1$, $y=-7$; (ii) $x=0$, $y=-5$. Which is the greater -7 or -8 ?

28. Simplify $\frac{a}{-b}-\frac{-a}{b}+\frac{-a}{-b}$.

29. If $x+y=1$, what is y when $x=6$? What is x when $y=2x$?

30. What can you say about x, if $2x$ is greater than $3x+5$?

31. What can you say about x if $x^2=36$?

32. What can you say about x if $-3x-2$ is positive ?

33. *Sketch* the graph of $-2-x$ for values of x from 4 to -4.

34. For what value of x is $\frac{x+1}{2}$ equal to $\frac{x-1}{3}$?

35. The average of three numbers is b; one of them is $-a$, another is 0; what is the third ?

36. What can you say about x if the square of $x-4$ is equal to the square of $x-7$?

37. If $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ and $u=\frac{f}{3}$, express v in terms of f.

38. What is T if $T°$ Fahrenheit is the same temperature as $(-T°)$ Centigrade ? [Use the formula VII. *f*. No. 6.]

39. If $u+v=0$ and $u-v=1$, what is the value of uv ?

40. If x is greater than y, does it follow that ax is greater than ay ? Consider the values $a=2, \frac{1}{2}, 0, -\frac{1}{2}, -2$.

41. Is $(x+3)^2$ necessarily greater than $(x+1)^2$?

42. What is x if $(x-5)^2=16$? [Two answers.]

43. What is y if $(y+4)^2=25$? [Two answers.]

44. What is b, if $\frac{1}{b}=-2$?

45. What is (i) the term of degree 2, (ii) the coefficient of y, i) the constant term in (a) y^3-3y^2-y+1, (b) $1-(y^2+2y+4)$?

46. A stone, thrown vertically upwards from the top of a tower, rises $(40t-16t^2)$ feet in t seconds. It strikes the ground after 4 seconds. How high is the tower ?

47. If the temperature of the air is $T°$ Centigrade, the velocity of sound is $\frac{5}{2}(436+T)$ feet per second. What is the temperature, when sound travels at 1080 feet per second ?

48. Fill in the blanks in the following:

(i) $12x-18y=(-6)[\quad]=6x-[\quad]$.
(ii) $b-a=(-1)[\quad]=2b-[\quad]$.

CHAPTER VIII

LINEAR RELATIONS

There is no difficulty in learning mechanical rules for the solution of simultaneous equations, but some introductory work is essential if the pupil is to appreciate properly the meaning of the process and of the results, when obtained. By the application of graphical methods to practical examples, the pupil is led to realise, of necessity slowly and gradually, what is meant by functionality. But further, the graphical treatment concentrates attention on the fact that in simultaneous equations we are looking for a value of the variable for which two functions are equal and, when this has been realised, leads to the more difficult idea of satisfying simultaneously (when possible) two relations between two variables. After the principles have been grasped, there must of course be practice in mechanical application : it is convenient to reserve this for a separate chapter.

Oral Example I. A has £180 in the bank at the present time and his savings enable him to increase his bank balance steadily at the rate of £30 each year. If his balance is £b after n years from now, what is the formula for b in terms of n ?

Make a table of values, direct from the data *and* by using your formula ; verify the fact that the two methods agree.

n	0	1	2	3	-1	-2
b	180					

Represent these results graphically.

What kind of a graph do you get ? Is it what you would expect, if so, why ?

How does the graph slope ?

The graph corresponds to the formula, $b=180+30n$. What is there about this equation that cause the upward slope ?

Read off from the graph the value of b when $n=-4$ and $n=8$.

Read off from the graph the value of n, if $b=300$.

Next suppose that C has £600 in the bank at the present time and that his expenditure causes him to decrease his bank balance

steadily at the rate of £40 each year. If his balance is £b after n years from now, what is the formula for b in terms of n?

Make a table of values, by two methods as above.

n	0	1	2	3	-1	-2
b	600					

Represent these results graphically, with the same axes and units as before.

What kind of a graph do you get? Is it what you would expect?

How does the graph slope?

The graph corresponds to the formula $b = 600 - 40n$. What is there about this equation that causes the downward slope?

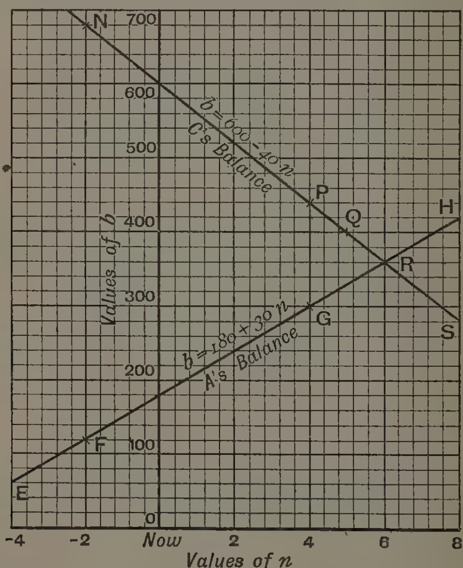

Fig. 173.

Which slope is numerically the greater and why?

Read off from the graph the value of b when $n = 4$ and $n = 8$.

Rad off forme the graph the value of n if $b = 400$.

Which has the larger balance after 5 years, after 8 years?

Which has the larger balance, and by how much, when $n=4$ and when $n=0$?

Interpret the points F and N shown on the graphs.

What does the point R represent?

Since A's balance increases steadily and C's balance decreases steadily, there must come a moment when their balances are equal. When is this? And how much is this balance?

For what value of n is the b of A's graph equal to the b of C's graph?

Lastly, take the two equations $b=600-4n$, $b=180+3n$ and find the value of n which gives equal values of b in the two cases, *i.e.* solve $600-4n=180+3n$.

Then find the corresponding value of b.

Note. If we regard A's balance as increasing steadily all the time, the balance at any one moment is represented by a definite point on the first graph; conversely each point on that graph gives us his balance at some particular moment.

Each point on the graph therefore supplies one pair of values of n and b, which fits the equation $b=180+3n$; there are, of course, an unlimited number of such pairs.

In the same way, every point on the graph of C's balance gives us one pair of values of n and b which fits the equation $b=600-4n$.

The point R where the two graphs cut gives us the unique pair of values of b and n which fits each of the equations $b=600-4n$, $b=180+3n$. The value of n obtained in this way gives the number of years that pass, up till the moment when the two bank balances become equal; and the corresponding value of b gives the amount of the common bank-balance. At this moment, and only at this moment, the two equations apply both to A and to C and we call them **simultaneous equations.**

If the equations $\begin{cases} b=600-4n \\ b=180+3n \end{cases}$ are simultaneous, then

$$600-4n=180+3n \quad \text{or} \quad 7n=600-180=420;$$

$$\therefore\ n=60.$$

And $b=180+3n=180+3\times 60=180+180=360.$

To check this result, take the other equation, then

$$b=600-4n=600-4\times 60=600-240=360.$$

If two linear relations between two variables can be satisfied simultaneously, the values of the variables which fit both relations can always be found by the substitution method, as above.

Exercise VIII. *a*.

1. Two tanks A and B contain at the present moment 120 gallons and 300 gallons of water respectively. Water is running into A at the rate of 10 gallons per minute and is running out of B at the rate of 20 gallons per minute. If A

contains n gallons after t minutes, find a formula for n in terms of t. Make a table of values and draw a graph showing how much water A contains during the first 10 minutes.

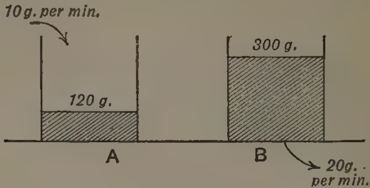

FIG. 174.

If B contains n gallons after t minutes, find a formula for n in terms of t. Make a table of values and draw a graph showing how much water B contains during the first 10 minutes. Where do the two graphs cut and what does this mean ?

Take the two expressions obtained for n and equate them and find t and then the corresponding value of n. What does this mean ?

2. Winchester is 50 miles from Oxford. A leaves Winchester at 10 a.m. and drives to Oxford at 20 miles an hour. B leaves Oxford at 11 a.m. and drives to Winchester at 30 miles an hour. Suppose that A is d miles *from Oxford* at t hours after 10 a.m.; find the formula for d in terms of t.

Make a table of values for the relation between d and t in A's journey and draw the graph for values of t from 0 to $2\frac{1}{2}$.

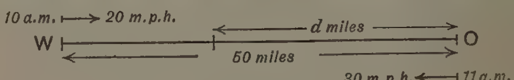

FIG. 175.

Suppose that B is d miles from Oxford at t hours after 10 a.m.; find the formula for d in terms of t; make a table of values for B's journey and draw the graph for values of t from 1 to $2\frac{1}{2}$. Where do the graphs cut and what does this mean ?

Take the two expressions obtained for d and equate them and find t and then the corresponding value of d. What does this mean ?

3. Two unlike spiral springs A and B when unstretched are of lengths 20 in. and 14 in. respectively. Each is suspended

from one end and bodies of various weights are attached at the other end. A stretches $\frac{1}{2}$ inch for each 1 oz. weight attached; B stretches 5 inches for each 4 oz. weight attached. Let the total length of a spring be l in. when the total weight attached is W oz. Write down the formula for l in terms of W for each spring and proceed as in questions 1 and 2.

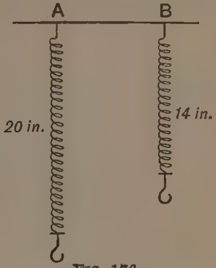

FIG. 176.

4. Compare the two following rules for income-tax:

Rule I. No tax on the first £200 of a man's income and 5s. in the £ on the rest.

Rule II. No tax on the first £160 of a man's income and 4s. in the £ on the rest.

Suppose the tax on an income of £I is £T; make tables of values for I and T for the two rules for values of I from 200 to 500, and represent the rules graphically. Find the formula for T in terms of I in each case. Find graphically and by calculation the income for which the two rules give the same tax.

Find, by the substitution method, pairs of values which satisfy the following simultaneous relations:

5. $y=7-2x,\quad y=x-2.$

6. $y=3+x,\quad y=12-x.$

7. $x=2y-3,\quad x=7y-13.$

8. $u=v-3,\quad u=5v-19.$

9. $c=5-3b,\quad c=13+b.$

10. $P=3Q,\quad P=20-2Q.$

11. $\dfrac{x}{y}=3,\quad x=15-2y.$

12. $s-3t=2,\quad s+t=14.$

13. $b=\frac{1}{3}(a-1),\quad b=\frac{1}{2}(a+1).$

14. $q=r-3,\quad q=10r-30.$

15. $y=\dfrac{x}{6}+\dfrac{1}{3},\quad y=\dfrac{x}{5}.$

16. $\dfrac{x}{y}=2\frac{1}{2},\quad \dfrac{x}{y+1}=2.$

We represented in Fig. 173 two linear relations and then found from the graph the pair of values of the variables, common to them. We can use this method in all cases.

Example II. Have the relations $3x+2y=11$, $2x+5y=18$ a common pair of values of x and y? If so, find them.

If $3x+2y=11$, then $2y=11-3x$.

$$\therefore\ y=\frac{11-3x}{2}.$$

We can use this form of the relation to obtain a table of values.

$x=0$	1	2	3	4
$y=5{\cdot}5$	4	$2{\cdot}5$	1	$-0{\cdot}5$

Similarly, if $2x+5y=18$, $5y=18-2x$.

$$\therefore\ y=\frac{18-2x}{5}.$$

This leads to the following table of values.

$x=0$	1	2	3	4
$y=3{\cdot}6$	$3{\cdot}2$	$2{\cdot}8$	$2{\cdot}4$	2

The result of plotting these values is shown in Fig. 177.

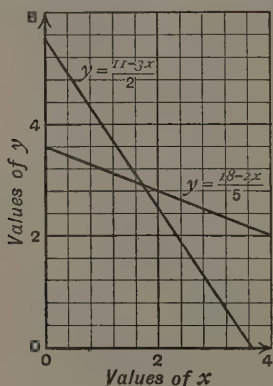

FIG. 177.

The point of intersection of the two graphs corresponds to the pair of values, $x \doteq 1{\cdot}7$, $y \doteq 2{\cdot}9$. This is therefore the common pair of values of the two linear relations.

The result may also be obtained by using the substitution method.

As before, the relations $3x+2y=11$, $2x+5y=18$ may be written in the form:

$$y=\frac{11-3x}{2} \quad \text{and} \quad y=\frac{18-2x}{5}.$$

The value of x which gives equal values of y for the two relations must therefore satisfy the equation

$$\frac{11-3x}{2}=\frac{18-2x}{5}.$$

$$\therefore\ 5(11-3x)=2(18-2x) \quad \text{or} \quad 55-15x=36-4x;$$

$$\therefore\ 4x-15x=36-55 \quad \text{or} \quad -11x=-19;$$

$$\therefore\ 11x=19; \quad \therefore\ x=\frac{19}{11}.$$

If $x=\frac{19}{11}$

$$y=\frac{11-3\times\frac{19}{11}}{2}=\frac{121-57}{22}=\frac{64}{22}=\frac{32}{11}.$$

$$\therefore\ x=\tfrac{19}{11}\simeq 1{\cdot}73, \quad y=\tfrac{32}{11}\simeq 2{\cdot}91$$

is the required common pair of values for the two relations.

Exercise VIII. *b*.

1. Represent graphically the functions $5-x$ and $3x-1$; hence find a common pair of values satisfying the relations $y=5-x$, $y=3x-1$. Compare the result with that obtained from the substitution method.

Proceed as in No. 1 in the following examples, Nos. 2-7.

2. $y=2x-1$, $y=9-2x$.	3. $y=5x+4$, $y=2x+9$.	4. $y=\frac{1}{3}(x-2)$, $y=\frac{1}{4}(3-x)$.
5. $y=3x$, $y=7-2x$.	6. $y=\frac{1}{4}(3-5x)$, $y=\frac{x}{4}+3$.	7. $y=\frac{1}{3}(2x+6)$, $y=-\frac{1}{2}(x+5)$.

Find y in terms of x in the following examples, Nos. 8-16.

8. $2x+y=7$.	9. $3x-y=11$.	10. $5x+2y=12$.
11. $\frac{x}{2}+3y+1=0$.	12. $\frac{x}{3}-4y=12$.	13. $\frac{x}{2}+\frac{2y}{3}=10$.
14. $x=\frac{2y-3}{5}$.	15. $\frac{x}{5}+\frac{y}{6}+1=0$.	16. $\frac{x}{y-1}=\frac{3}{4}$.

17. Represent graphically the relations

$$3x+2y=6, \quad 4y-4x=7,$$

and find a common pair of values of x and y satisfying them. Compare the result with that obtained from the substitution method.

18. Represent graphically the relations $3x+y=5$, $6x+2y=7$. Can you find a common pair of values of x and y satisfying them?

Use the substitution method to find a pair of values satisfying the following simultaneous relations:

19. $a+2b=11$, $2a-b=2$.

20. $3p-q=11$, $2p-3q=5$.

21. $3x-7y=35$, $2x+5y=4$.

22. $\frac{x}{2}+\frac{y}{3}=3$, $2x+3y=2$.

23. $\frac{u}{5}+2v=9$, $\frac{u}{2}-\frac{v}{6}=7$.

24. $3P=4Q$, $\frac{1}{2}P+\frac{2}{5}Q=4$.

25. $3y-4z+2=0$, $3z+y+1=0$.

26. $\frac{1}{2}c-\frac{2}{3}d+3=0$, $\frac{1}{2}c-\frac{3}{5}d+1=0$.

EASY REVISION PAPERS. A 16-20

A. 16

1. If a third-class fare at $1\frac{1}{2}$d. per mile from A to B is $6p$ shillings and sixpence, what is the first-class fare at $2\frac{1}{2}$d. per mile?

2. Simplify (i) $\frac{a}{b}\times ab$; (ii) $r-\frac{2r}{3}$; (iii) $\sqrt{\left(x^2-\frac{3x^2}{4}\right)}$.

3. Simplify $2(x+y)-4(x-y)$.

Multiply $(b-a)$ by $-ab$ and subtract the result from a^3-b^3; arrange the answer in descending powers of a.

4. Express the following statement as an equation, without simplifying: the difference between the squares of two consecutive positive integers is 39.

5. Solve $x=2(12-2x)$; explain each step and check your answer.

A. 17

1. The base of a triangle is $2e$ inches, the height is $1{\cdot}7e$ inches; what is its area? If another altitude is e inches, find one other side.

2. What are the values of (i) $(+3)+(-5)$; (ii) $(-5)+(+3)$; (iii) $(+1)-(-2)$; (iv) $(-6)-(+2)$; (v) $(-6)\times(-2)$?

3. A journey of 150 miles takes me 5 hours; part of the time I am in a train travelling at 40 m.p.h. and the rest of it in a car travelling at 25 m.p.h. How far did I go by train?

4. Solve: $y=\frac{1}{5}(7x-4)$; $y=2(x-1)$.

5. In Fig. 178, $ABCD$ is a rectangle; the path PBQ is half the distance of the path $PADCQ$; how far is Q from B?

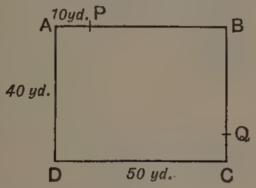

FIG. 178.

A. 18

1. By how much per cent. is $5k$ greater than $4k$?

2. Simplify (i) $\dfrac{2a}{2ab}$; (ii) $\dfrac{2ab}{2a}$; (iii) $\dfrac{b}{\frac{1}{a}}$; (iv) $\dfrac{a}{b+\frac{1}{c}}$.

3. If the radius of the base of a cone is r in. and its height is h in., the volume may be taken as $\frac{22}{21}r^2h$ cu. in.; find in cu. in. to two significant figures the volume of a cone of height 10 in. and base diameter 3 ft.

4. The following table gives the number of shipbuilders per 1000 out of work in different months of a year:

Jan., 80; Feb., 68; Mar., 61; Ap., 63; May, 60; June, 62; July, 61; Aug., 68; Sept., 81; Oct., 92; Nov., 99; Dec., 98.

Represent this information graphically and describe the general features of the unemployment.

5. A tank which holds 900 gallons is filled by two pipes, one of which can supply 40 more gallons per minute than the other. When both pipes are turned on, the tank is filled in ten minutes. How long would the smaller pipe take to fill the tank by itself?

A. 19

1. Find in pence the simple interest on £P for a month at 5 per cent.

2. What is the reciprocal of (i) a; (ii) $\frac{1}{b}$; (iii) $\frac{a}{b}$; (iv) $1+\frac{p}{q}$.

3. Write down the value of (i) $-a+4a$; (ii) $-a-(-4a)$; (iii) $(-6a)\div(-2a)$; (iv) $10a\div(-5a)$.

4. Solve: (i) $3(7-x)+2(5-x)=1$;
(ii) $y=9-7x$, $y=\frac{1}{7}(15-x)$.

5. One car takes 7 hours over a journey which another car, travelling 5 miles per hour faster, does in 6 hours. What is the length of the journey ?

A. 20

1. A right angle is divided into two parts; one part is $(45+x)$ degrees; by how much does it exceed the other part ?

2. Add together $1-2x+x^2$ and $2x^2-x-1$, and subtract the result from $3x-2$.

3. If $2x+y=z$, what is x when (i) $y=1$ and $z=7$; (ii) $y=7, z=1$; (iii) $y=1, z=-1$; (iv) $y=-4, z=-8$?

4. A first-class ticket costs p pence per mile and a third-class ticket q pence per mile. What is the cost in shillings of 3 first-class and 1 third-class ticket from London to Carlisle (300 miles) ?

5. I cycle at a steady rate for 2 hours and then increase my speed for half an hour by 2 miles an hour: at the end of that time I have gone 24 miles At what rate did I start ?

CHAPTER IX

SIMULTANEOUS EQUATIONS AND PROBLEMS

Generalised Arithmetic.

THE pupil has probably already met in Arithmetic a certain number of problems involving two unknowns. It is suggested that some examples of this kind, given below, should be taken orally.

Example I. The daily wages of 3 men and 2 boys amount to £1 15s.; and the daily wages of 3 men and 5 boys amount to £2 7s. Find the daily wage of a man and of a boy.

The daily wages of 3 men and 5 boys amount to 47s.
The daily wages of 3 men and 2 boys amount to 35s.

$\therefore$ the daily wages of 3 boys amount to $(47-35)$s. $=12$s.
$\therefore$ the daily wage of 1 boy is $\frac{12}{3}$s. $=4$s.
$\therefore$ the daily wages of 3 men anount to $(35-8)$s. $=27$s.
$\therefore$ the daily wage of 1 man is $\frac{27}{3}$s. $=9$s.

Note. We have solved this problem by obtaining, from the two given statements, a single statement involving only the daily wage of a boy. In other words, we have *eliminated* the daily wage of a man.

Example II. The daily wages of 3 men and 2 boys amount to £2 6s.; and the daily wages of 1 man and 4 boys amount to £1 12s. Find the daily wage of a man and of a boy.

The daily wages of 3 men and 2 boys amount to 46s.
The daily wages of 1 man and 4 boys amount to 32s.
We wish now to eliminate either a man's wages or a boy's wages.
From the first statement we can say:
The daily wages of 6 men and 4 boys amount to 92s.
But the daily wages of 1 man and 4 boys amount to 32s.

$\therefore$ the daily wages of 5 men amount to $(92-32)$s. $=60$s.
$\therefore$ the daily wage of 1 man is $\frac{60}{5}$s. $=12$s.
$\therefore$ the daily wages of 2 boys amount to $(46-36)$s. $=10$s.
$\therefore$ the daily wage of 1 boy is 5s.

Note. Here, we have eliminated the boy's wages by altering the form of one of the given statements.

The work of this example may be expressed more shortly by using symbols.

Suppose that the daily wage of a man is m shillings and that the daily wage of a boy is b shillings.

Then $$3m+2b=46$$
and $$m+4b=32.$$

From the first relation, we have
$$6m+4b=92.$$
But $$m+4b=32;$$
$$\therefore\ 5m=60 \text{ or } m=12.$$

Substitute for m, then $3\times12+2b=46$;
$$\therefore\ 2b=46-36=10; \quad \therefore\ b=5.$$

Exercise IX. *a*.

Solve Nos. 1-5 first of all, without using symbols.

1. 5 lb. of apples and 2 lb. of pears cost 4s.; 1 lb. of apples and 2 lb. of pears cost 2s. Find the cost per lb. of apples and pears.

2. A gramophone with 20 records costs £8 10s.; the same gramophone with 50 records costs £12 5s.; find the price of the gramophone by itself, and the price of each record.

3. 6 lb. of potatoes and 5 lb. of tomatoes cost 2s. 5d.; 6 lb. of potatoes and 2 lb. of tomatoes cost 1s. 5d. Find the cost per lb. of potatoes and tomatoes.

4. A farmer finds that he can buy 3 cows and 5 sheep for £90 or he can buy 4 cows and 10 sheep for £140. Find the price of a cow and of a sheep.

5. A builder requires 3 lorry loads and 8 cart loads to fetch 15 tons of gravel: he would require 2 lorry loads and 20 cart loads to fetch 21 tons of gravel. What amount of gravel is there in a lorry load and in a cart load?

6. With the data of No. 1, suppose that 1 lb. of apples costs a pence and that 1 lb. of pears costs p pence. Re-write the given statements, using symbols, and so find the values of a, p.

7. With the data of No. 2, suppose that the gramophone costs £g, and that 1 record costs r shillings. Re-write the given statements, using symbols, and so find the values of g, r.

8. With the data of No. 3, suppose that 1 lb. of potatoes costs p pence and that 1 lb. of tomatoes costs t pence. Re-write the given statements, using symbols, and so find the values of p, t.

9. With the data of No. 4, suppose that 1 cow costs £c, and that 1 sheep costs £b. Re-write the given statements, using symbols, and so find the values of c, b.

10. With the data of No. 5, suppose that a lorry load of gravel weighs l tons and that a cart load of gravel weighs c tons. Re-write the given statements, using symbols, and so find the values of l, c.

Solution by Elimination.

Example III. Obtain from the equations :

$$4x + 11y = 5, \quad 10x - 7y = 3,$$

an equation which does not contain x.

Multiply each side of the first equation by 5 ;

$$\therefore \quad 20x + 55y = 25.$$

Multiply each side of the second equation by 2 ;

$$\therefore \quad 20x - 14y = 6.$$

Subtract, then $\qquad 69y = 19.$

Note. (1) This process may be described as *eliminating* x from the two given equations.

(ii) If the coefficients of x in the two relations are numerically equal and of the same sign, we can eliminate x by subtracting ; if they are numerically equal but of opposite sign, we can eliminate x by adding.

Example IV. Solve the equations :

$$5x + 6y = 16, \quad \text{.................................(1)}$$

$$3x - 4y = 2. \quad \text{.................................(2)}$$

Multiply each side of equation (1) by 2 and each side of equation (2) by 3.

$$\therefore \quad 10x + 12y = 32$$

and $\qquad 9x - 12y = 6.$

Adding, we have $\qquad 19x = 38$;

$$\therefore \quad x = 2.$$

Substitute for x in equation (1) ;

$$\therefore \quad 10 + 6y = 16 ;$$

$$\therefore \quad 6y = 6 ;$$

$$\therefore \quad y = 1.$$

$\therefore$ the solution is $\mathbf{x = 2,\ y = 1}$.

Check the answer by substituting in equation (2):

$$\text{left side} = 3x - 4y = 6 - 4 = 2 = \text{right side}.$$

Note. (i) The process of obtaining from two equations in x, y a single equation which does not contain y is called "eliminating y." It does not matter whether you eliminate y or eliminate x; always do whichever is easier.

(ii) When you have found one unknown, it does not matter in which equation you substitute to find the other. Take the *easiest* equation that has occurred in the working. But when checking the answer, *do not substitute in the same or an equivalent equation*; for checking you should always use one of the *original* equations.

Exercise IX. *b*.

Solve the following pairs of simultaneous equations and check your answers:

In each case, write down *at the start* which unknown can be eliminated the more easily, or whether it makes no difference.

1. $x + y = 17,\quad x - y = 3.$
2. $2x + y = 11,\quad x + y = 7.$
3. $3x + y = 10,\quad x - y = 2.$
4. $u - 2v = 12,\quad u + 3v = 4.$
5. $3p + q = 11,\quad 7p + q = 3.$
6. $p + 2q = 0,\quad p + 5q = 12.$
7. $y = 2z + 1,\quad y = 3z - 2.$
8. $z - 2x = 0,\quad 3z + 2x = 6.$
9. $3t + 7v = 12,\quad t + 7v = 20.$
10. $2x + y = 9,\quad x - 3y = 1.$
11. $2x - 3y = 5,\quad x + 2y = 6.$
12. $a - 3b = -1,\quad 2a - 4b = 2.$
13. $5t - 2z = 1,\quad 2t + z = 13.$
14. $u - 2v = 0,\quad 4u + 3v = 11.$
15. $y = 2z,\quad 4y - z = 22.$
16. $x + 3z = 2,\quad 4x + 5z = 6.$
17. $6x - 2y = 5,\quad 4x + 7y = -5.$
18. $5u - 7v = 20,\quad 9u - 11v = 44.$
19. $3x + 7y = 4\tfrac{3}{4},\quad 2x + 3y = 2\tfrac{3}{4}.$
20. $3x - 4y = 2,\quad 2x = 3y.$
21. $3x + 7t - 7 = 0,\quad 5x + 3t + 36 = 0.$
22. $12x = 4 + 8y,\quad 10y = 3 - x.$
23. $50x - 20y = 10,\quad 91x - 26y = 39.$
24. $5p - 2q = 9\tfrac{1}{2},\quad 2p + 3q = 9\tfrac{1}{2}.$
25. $5(x-1) + 2(y+1) = 42,\quad x = 2y - 3.$
26. $3(x+2) - 4(y-2) = 9,\quad 5(x+2) + 2(y-2) + 3 = 0.$

27. $x-3=y-5=10.$ 28. $2x+y=3x-y=15.$

29. $x-y=y-x+2=3y.$

30. $x+5y-1=3x+2y+1=x+y+7.$

Easy Problems.

Example V. A certain sum of money is sufficient to pay A's wages for 20 days *or* B's wages for 30 days. When A receives a rise of 3s. per day and B a rise of 1s. per day, the same sum will just pay the joint wages of A and B for 10 days. Find the original wages of each.

Let A's original wages be a shillings per day.

Let B's original wages be b shillings per day.

Then the sum of money is $20a$ shillings and also $30b$ shillings.

$$\therefore\ 20a=30b.$$

Divide by 10, $\therefore\ 2a=3b$ or $2a-3b=0.$ ....(1)

A's new wages are $(a+3)$ shillings per day, B's new wages are $(b+1)$ shillings per day.

$\therefore$ the sum of money is also $10(a+3+b+1)$ shillings.

$$\therefore\ 20a=10(a+3+b+1).$$

Divide by 10, $\therefore\ 2a=a+b+4$ or $a-b=4.$ ....(2)

Multiply by 3, $\therefore\ 3a-3b=12.$

But from (1), $2a-3b=0.$

Subtract, $\therefore\ a=12.$

Substitute in (2), $\therefore\ 12-b=4$ or $-b=-8$;

$$\therefore\ b=8;$$

$\therefore$ originally A receives 12s. per day and B receives 8s. per day.

Check. In 20 days, A receives 12×20s. $=240$s.

In 30 days, B receives 8×30s. $=240$s.

After the rise, A receives 15s. per day, B receives 9s. per day; their joint wages are $9+15=24$s. per day.

In 10 days, they receive together 24×10s. $=240$s.

Exercise IX. *c*.

Solve the following by means of *simultaneous* equations.

1. The sum of two numbers is 25 and their difference is 3; find them.

2. The perimeter of a rectangular sheet of paper is 30 inches; and its length is 3 inches more than its breadth; find the length.

3. The external perimeter of the network in Fig. 179 is 20 in., and the total length of all the lines forming it is 3 ft.; find the external length and breadth.

FIG. 179.

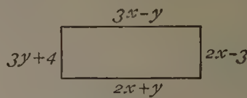

FIG. 180.

4. The number of boys in a school is 450, and there are 30 more on the classical than on the modern side; find the number on each side.

5. Find the length and breadth of the rectangle in Fig. 180, the units of the data being inches.

6. A bench 18 ft. long holds either 3 men and 8 boys or 6 men and 4 boys. What length of bench is required for 10 men and 10 boys?

7. A penny weighs x oz. and a halfpenny weighs y oz.; find x, y, given that three pennies weigh as much as five halfpennies and that 12 pennies with 4 halfpennies weigh 4·8 oz.

8. A light rod is supported by loops attached to two spring balances; and a heavy load W lb. is attached to the rod as shown in Fig. 181. If the readings on the spring balances are P lb., Q lb., it is known that $P+Q=W$ and $a \,.\, P=b \,.\, Q$.

(i) Find P and Q, if $W=20$, $a=10$, $b=6$.

(ii) Find P and W, if $Q=9$, $a=8$, $b=12$.

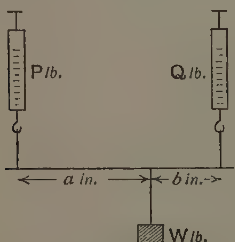

FIG. 181.

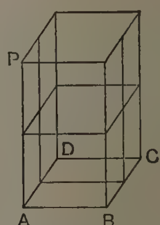

FIG. 182.

9. A man wishes to fetch 55 tons of gravel from a pit; he requires 16 lorry loads and 20 cart loads or 10 lorry loads and 40 cart loads. What is the weight of a cart load?

10. A skeleton cuboid (see Fig. 182) has a square base

ABCD; the wire composing the cuboid is 11 ft. long, and of this, 7 ft. is used for the outside edges, *i.e.* all the rims. Find the dimensions of the cuboid.

11. A pamphlet contains 10 full pages of print, some in small and some in large type; when small type is used, a page takes 550 words, but with large type only 350 words. There are 4600 words in all; how many pages are printed in large type?

12. A piece of wire $6\frac{1}{2}$ ft. long can be bent into either of the right-angled cornered shapes in Fig. 183, the units of the data being inches. Find x and y.

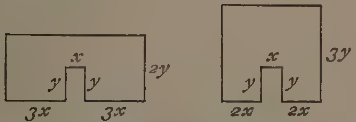

FIG. 183.

13. A try counts 3 points and a goal 5 points: one side scored x tries and y goals, the other $(x-3)$ tries and $2y$ goals; the latter won by 26 points to 25 points; find the tries and goals scored by each side.

14. In Fig. 184, $\angle BAC$ is $(2x-y)$ degrees and also $(2y+x)$ degrees. Find numerically the size of $\angle BAC$.

15. In Fig. 184, $\angle BAC = 2\angle ABC$ and $\angle ACB - \angle ABC = 36°$; find the angles of the triangle.

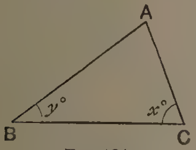

FIG. 184.

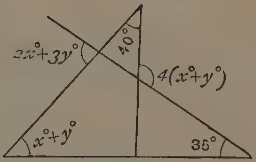

FIG. 185.

16. Find x and y in Fig. 185, the angles being all measured in degrees.

17. (i) In a number of two digits, the unit digit is x and the tens digit is y. Express the value of the number in terms of x and y.

(ii) A number of two digits is equal to 21 times the difference between the digits: also the number formed by reversing the digits is 36 less than the original number. What is the original number?

18. *ABCD* is a chain, 5 ft. long, hanging over a beam 8 inches wide (see Fig. 186). The chain will slip if one-quarter of the difference of the lengths of the two vertical portions is greater than the length of the horizontal portion. What is the greatest possible length of *CD*, if the chain does not slip?

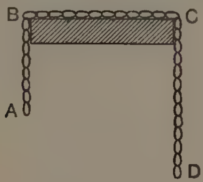

FIG. 186.

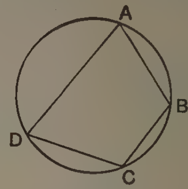

FIG. 187.

19. In Fig. 187, $\angle A = (2x+20)$ degrees, $\angle B = 3(y+5)$ degrees, $\angle C = 2(y+15)$ degrees, $\angle D = 2x$ degrees; find x and y.

Example VI. Solve the equations:

$$\frac{x}{5} - 2y = 4\tfrac{1}{6}, \quad \text{.........................} (1)$$

$$\frac{x}{2} + \frac{3y}{4} = -3. \quad \text{.........................} (2)$$

Multiply each side of equation (1) by 5 and each side of equation (2) by 4;

$$\therefore \; x - 10y = 20\tfrac{5}{6} \quad \text{.........................} (3)$$

and $$2x + 3y = -12. \quad \text{.........................} (4)$$

[Here, it is easier to eliminate x than y.]

Multiply each side of equation (3) by 2;

$$\therefore \; 2x - 20y = 41\tfrac{2}{3}. \quad \text{.........................} (5)$$

Subtract equation (5) from equation (4):

$$\therefore \; 23y = -12 - 41\tfrac{2}{3} = -53\tfrac{2}{3};$$

$$\therefore \; y = -\tfrac{161}{3} \times \tfrac{1}{23} = -\tfrac{7}{3} = -2\tfrac{1}{3}.$$

Substitute for y in equation (3):

$$\therefore \; x - 10(-\tfrac{7}{3}) = 20\tfrac{5}{6};$$

$$\therefore \; x + \tfrac{70}{3} = 20\tfrac{5}{6};$$

$$\therefore \; x = 20\tfrac{5}{6} - 23\tfrac{1}{3} = -3 + \tfrac{5-2}{6} = -3 + \tfrac{1}{2};$$

$$\therefore \; x = -2\tfrac{1}{2}.$$

$\therefore$ the solution is $\mathbf{x} = -2\tfrac{1}{2}$, $\mathbf{y} = -2\tfrac{1}{3}$.

Check these values by substituting in equation (2).

Note. (i) If x or y have fractional coefficients, the first step usually is to clear these terms of fractions; you need not remove fractions (or decimals) from the terms which do not contain x or y.

(ii) Always consider whether x or y is the easier to eliminate.

(iii) In substituting, choose the easiest equation; this is usually one from which fractions have been removed. But when checking, choose one of the *original* equations and choose that one which is not connected with the equation in which you have substituted; *e.g.* in the work above, (3) is derived from (1), so check in (2), not in (1).

EXERCISE IX. *d.*

Solve the following pairs of simultaneous equations:

1. $\frac{x}{2}-2y=5,\quad \frac{x}{2}+y=8.$
2. $x-\frac{5y}{2}=-2,\quad x-2y=-1\frac{1}{2}.$
3. $\frac{1}{2}x-\frac{1}{3}y=3,\quad \frac{1}{4}x-5y=1\frac{1}{2}.$
4. $\frac{3x}{7}+2y=4,\quad \frac{5x}{7}+3y=7.$
5. $\frac{3x}{4}+\frac{5y}{6}=3\frac{1}{6}.\quad \frac{2x}{5}-\frac{y}{4}=1\frac{11}{20}.$
6. $x=\frac{1}{3}(y-4).\quad y=\frac{1}{2}(x-6).$
7. $x+y=3\frac{1}{14}.\quad x-y=2\frac{1}{12}.$
8. $\frac{5x}{2}-\frac{4y}{3}=7.\quad \frac{x}{4}+\frac{y}{3}=3\frac{1}{2}.$
9. $\frac{1}{2}(x-2)=\frac{1}{3}(y-3).\quad x+y=10.$
10. $3x+y=\frac{1}{44}.\quad 3x-2y=1\frac{1}{55}.$
11. $x+y=17,\quad \frac{x}{y}=\frac{2}{3}.$
12. $\frac{x}{y}=\frac{3}{4},\quad \frac{x-5}{y-5}=\frac{7}{11}.$
13. $\frac{x+1}{y-2}=\frac{2}{3}.\quad \frac{x-1}{y+2}=\frac{4}{7}.$
14. $\frac{2x+5}{3y+7}=\frac{2x}{3y}.\quad x-y=2.$
15. $\frac{y+4}{3}=\frac{x+1}{4}.\quad \frac{7}{x}=\frac{2}{y}.$
16. $0{\cdot}3x+0{\cdot}2y=4,\quad 2{\cdot}5x-y=12.$
17. $8x+y=4{\cdot}3,\quad 3x-4y=0{\cdot}3.$
18. $1{\cdot}5x-2y=4{\cdot}5,\quad 0{\cdot}6x-2{\cdot}5y=-8{\cdot}4.$
19. $0{\cdot}1x+1{\cdot}3y=8{\cdot}2,\quad 0{\cdot}7x+5y=32{\cdot}8.$
20. $x+5{\cdot}1y=3{\cdot}85,\quad x+7{\cdot}3y=12{\cdot}65.$
21. $0{\cdot}5x-0{\cdot}7y=2,\quad 0{\cdot}9x-1{\cdot}1y=4{\cdot}4.$
22. $x-\frac{y-1}{4}=8,\quad y-\frac{2x+1}{5}=-6.$

EXTRA PRACTICE EXERCISES. E.P. 8.

Simultaneous Equations.

Solve the following pairs of simultaneous equations :

1. $a+b=13,\quad a-b=4.$
2. $u-v=11,\quad u+v=1.$
3. $x-y=9,\quad x+y=0.$
4. $p+q=8,\quad q-p=3.$
5. $2l+m=12,\quad l=m.$
6. $2u-v=7,\quad u+v=8.$
7. $3x-2y=6,\quad x+3y=13.$
8. $7x-y=2,\quad 6x=y.$
9. $2x+5y=8,\quad 3x+4y=5.$
10. $y=2x+1,\quad 3y=5(x+1).$
11. $3x+4y=7,\quad y=0.$
12. $4x+3y=1,\quad 5x+4y=2.$
13. $l+m=2\frac{1}{3},\quad m-l=1\frac{1}{6}.$
14. $a+2b=5\frac{1}{4},\quad a-b=1\frac{1}{2}.$
15. $4x-3y=3,\quad y=2x.$
16. $t+2z=12,\quad z=4t.$
17. $c=3-2d,\quad d=12-2c.$
18. $3p+4q-3=0,\quad 2p-5q+5=0.$
19. $3x+5y=21,\quad x+2y=7.$
20. $2x+7y-22=0,\quad 3x+6y-15=0.$
21. $4x+3y+13=0,\quad 3x+2y+8=0.$
22. $2r+s+10=0,\quad 3r-2s+1=0.$
23. $3p+2z=4,\quad 4p+z+3=0.$
24. $2x+11y=12,\quad 3x-2y+19=0.$
25. $c-7=d+2=5.$
26. $2p-1=0=3+q.$
27. $x-3y-5=2x+y-3=0.$
28. $y=2(x+y-1)=-2(x+y+3).$
29. $2x+3y=x+1=4-y.$
30. $x+2y+3=4x+4y-1=3x+3y+2.$
31. $\frac{1}{2}p+\frac{1}{3}q=1.\quad 2p+3q+1=0.$
32. $\frac{x-1}{3}+\frac{y+1}{2}=1,\quad \frac{2x+1}{5}-\frac{3y+1}{4}=5.$
33. $\frac{1}{x}-\frac{1}{y}=2,\quad \frac{1}{x}+\frac{1}{y}=8.$
34. $\frac{3}{y}-\frac{1}{z}=1,\quad \frac{5}{y}+\frac{2}{z}=20.$
35. $\frac{2l-m}{3}=\frac{l-3m}{2},\quad \frac{l}{3}=1-m.$

36. $\dfrac{r-2s}{r}=\dfrac{3s}{2r},\ \dfrac{3}{r}-\dfrac{2}{s}=\dfrac{1}{rs}.$

37. $4x-2y=4y-5x=x+y-3.$

38. $2x-y-3=3y-x+4=5x-8y-4.$

39. $\dfrac{3p-7}{4}=\dfrac{2q+1}{3}=5.$

40. $\dfrac{y+2z}{2y-z}=\dfrac{3y-2}{z-7}=8.$

41. $\dfrac{x-y}{x+y}=\dfrac{3}{4}=\dfrac{x-1}{y+2}.$

42. $\dfrac{3x-1}{2y+1}=\dfrac{2x+2}{y+2}=4.$

PROBLEMS.

43. A knife and fork cost 6s.; the knife costs six pence more than the fork. What is the cost of each ?

44. A man distributes x oranges among n boys; if each receives 10 oranges, there are 3 left over; but there are four oranges too few for each boy to receive 11 oranges. Find x and n.

45. I am thinking of two numbers which when added make 90 and are such that one-third of the smaller is equal to one-seventh of the larger. What are they ?

46. 3 lb. of jam and 2 lb. of butter cost 8s.; also 6 lb. of jam and 3 lb. of butter cost 14s.; find the cost of 1 lb. of jam and of 1 lb. of butter.

47. 3 cows and 4 sheep cost £52; also 4 cows and 6 sheep cost £71; find the cost of one cow and of one sheep.

48. Can you find two numbers such that three times the smaller exceeds twice the larger by 3, and seven times the smaller exceeds five times the larger by 2 ?

49. A boy spends 4s. 6d., partly on the entrance money for an exhibition, and the rest on amusements inside; his brother, who spends twice as much on amusements, spends 7s. altogether. What did each spend on amusements and what was the entrance money ?

50. In 4 years' time, a father will be 3 times the age of his son; 4 years ago he was 5 times the age of his son. What are their present ages ?

51. I have to pay for 500 cigarettes two shillings more than for 3 lb. of tobacco: and I have to pay for 4 lb. of tobacco four shillings more than for 600 cigarettes. What do I pay for 1 lb. of tobacco and for 100 cigarettes ?

52. I wish to give one shilling each to some children, but find I have 1s. 6d. too little to do so. I therefore give them 10d. each and have one shilling over. How many children are there and how much money have I got with me ?

53. A owes £4, B owes £5 ; A could just pay his debt if he borrowed from B one-eighth of what B has ; B could just pay his debt if he borrowed from A two-sevenths of what A has. How much has each ?

54. Find two numbers such that the first is greater than half the second by 4, and three times the first is less than twice the second by 1.

55. If A gives B two pence, B has four times as much as A. If C gives A and B ten pence each, then B has twice as much as A. How much have A and B ?

56. A heap of half-crowns and florins is worth £2 10s. ; another heap, which contains only half as many half-crowns but twice as many florins, is worth £2 15s. How many coins of each kind are there in the first heap ?

57. Can you discover a fraction such that the result of either adding 1 to the numerator or of subtracting 3 from the denominator is equivalent to $\frac{1}{3}$? Is there more than one answer ?

58. A and B are two jugs ; an empty pail is just filled by 5 jugfulls from A and one from B or by 8 jugfulls from B. If 3 jugfulls from both A and B are put into it, there is still room for another 2 pints. How much does each jug hold ?

EXERCISE IX. e.

1. Two numbers, x and y, are in the ratio $7 : 11$; their difference is 36 ; find them.

2. Two numbers, x and y, are in the ratio $7 : 5$; twice the smaller exceeds the larger by 36. What are they ?

3. A shilling weighs $\frac{1}{5}$ oz. and a penny weighs $\frac{1}{3}$ oz. : 11 coins, some shillings, the rest pence, weigh 2·6 oz. What is the value of the coins ?

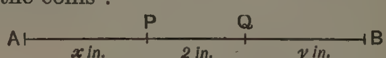

FIG. 188.

4. In Fig. 188, $\dfrac{AQ}{QB}=\dfrac{3}{2}$ and $\dfrac{AP}{PB}=\dfrac{1}{2}$; find the lengths of AP and QB.

5. A man travels 32 miles in $4\frac{1}{4}$ hours; part of it he walks at 4 m.p.h., and the remainder he bicycles at 10 m.p.h.; how far did he walk and how far bicycle?

6. x shillings and y half-crowns have the same value as $(y-1)$ shillings and $(x-2)$ half-crowns. If this value is £3, find x and y.

7. If A gives two-thirds of the money he has to B, then B will have $3\frac{1}{2}$ times as much as A. If B gives 9 shillings to A, then A will have $2\frac{1}{2}$ times as much as B. How much have they between them?

8. The resistance R lb. to a train of weight 100 tons running at V miles an hour is given by the formula $R=a+b \cdot V^2$, where a, b are constants. At 20 m.p.h., the resistance is 960 lb., and at 50 m.p.h. it is 2850 lb. Find the resistance at 30 m.p.h.

9. The tickets for a concert were priced at 2s. 6d. and 1s.; tickets to the value of £24 were sold. If the number who took shilling tickets were increased by 25 per cent. and the number who took half-crown tickets were decreased by 25 per cent., the takings would be £25. How many tickets of each kind were sold?

10. Find x and y in Fig. 189, the angles being measured in degrees.

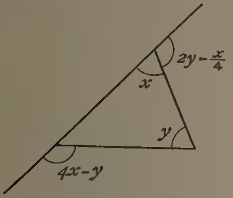

FIG. 189.

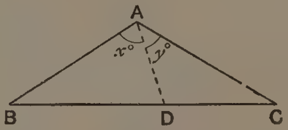

FIG. 190.

11. The incomes of two men are in the ratio 5 : 3; their expenditures are in the ratio 9 : 5. Each saves £30 a year. Find their incomes.

12. In Fig. 190, $AC=AB=BD$ and $\angle BAD=\frac{3}{4}\angle BAC$. Calculate x and y.

13. In a row of houses, the rents of the two end houses are £5 higher than those of the others; the total rent amounts to £190. If the rent of each end house is increased to double that of each other house, the total rent would amount to £220. Find the number of houses in the row

14. In Fig. 191, $AB = AC$ and BD bisects $\angle ABC$; the angles are all measured in degrees; find x and y.

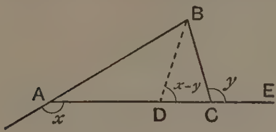

FIG. 191.

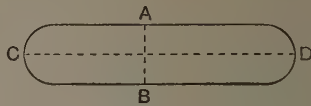

FIG. 192.

15. A race-course has the shape of a rectangle with semi-circular ends (see Fig. 192); the width AB is $2y$ yards and the maximum length CD is $2x$ yards. Express the length of one lap in terms of x, y, π. If $x = 3y$ and one lap is $\frac{1}{4}$ mile, find y. [Take $\pi = \frac{22}{7}$.]

Constants in Formulae.

The graphs in Chapter VIII. on p. 184 represent the relation between a man's bank-balance and the time, assuming that the balance is increasing or decreasing steadily with the time. Two special cases were examined:

$$\text{(i) } b = 180 + 30n;$$

$$\text{(ii) } b = 600 - 40n.$$

The constants in these relations fixed the shape and position of the graph. Each graph was a straight line, because in each case b was a function of n of the first degree.

In (i), the graph crosses the "now" axis at $b = 180$, because $b = 180$ when $n = 0$; and the slope of the line is fixed by the coefficient of n, viz. $+30$, the value of b *increases* by 30 units for each unit increase in n.

In (ii), the graph crosses the "now" axis at $b = 600$, because $b = 600$ when $n = 0$; and the slope of the line is fixed by the coefficient of n, viz. -40, the value of b *decreases* by 40 units for each unit increase in n.

The amount of any bank-balance which increases or decreases *steadily* with the time is given by a formula of this type,

$$b = B + A \cdot n,$$

where B, A are constants.

The same argument as was used before shows what these constants represent.

The graph corresponding to $b = B + A \cdot n$ crosses the "now" axis at $b = B$, because when $n = 0$, $b = B$. Therefore £B is the balance *now*; further, the slope of the line is fixed by the coefficient of n, viz. A, the value of b increases by A units for each unit increase in n. Of course, if A is negative, the balance decreases.

Example VII. A spiral spring is suspended from one end and various loads are attached in turn to the other end and the corresponding stretched lengths of the spring are measured and tabulated as follows :

Load in grams, w	100	200	400	500
Stretched length in cm., l	52·6	53·4	55	55·8

Investigate the formula connecting l and w.

By plotting the given observations we obtain the points marked in Fig. 193, and we see that these points lie on a straight line. Draw this line and produce it to cut the l-axis.

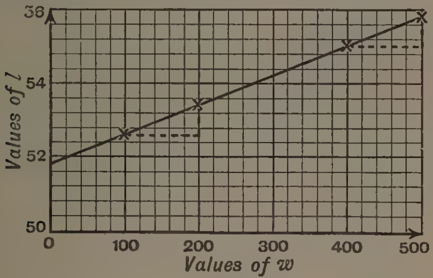

FIG. 193.

Since the graph is a straight line, the relation connecting l and w is of the form

$$l = B + A\,.\,w,$$

where A, B are constants to be determined.

From the graph, we see that the line crosses the l-axis at $l = 51{\cdot}8$, *i.e.* $l = 51{\cdot}8$ when $w = 0$;

$$\therefore\ B = 51{\cdot}8.$$

We also see that the slope of the graph is upwards and is such that l increases by 0·8 when w increases by 100 (see the dotted lines in Fig. 185) ; $\therefore$ l increases by 0·008 for each unit increase in w ;

$$\therefore\ A = +0{\cdot}008;$$

$\therefore$ the required formula is

$$l = 51{\cdot}8 + 0{\cdot}008w.$$

We may also obtain this result by calculation.

Taking the formula $l = B + Aw$.

Since $l = 52{\cdot}6$ when $w = 100$, we have

$$52{\cdot}6 = B + 100A.$$

Since $l=55$ when $w=400$, we have

$$55=B+400A.$$

Subtracting, $\quad 52{\cdot}6-55=100A-400A$;

$$\therefore\ 300A=55-52{\cdot}6=2{\cdot}4;$$

$$\therefore\ A=\frac{2{\cdot}4}{300}=0{\cdot}008.$$

And $\quad B=55-400A=55-400\times0{\cdot}008=55-3{\cdot}2=51{\cdot}8$;

$$\therefore\ \mathbf{l=51{\cdot}8+0{\cdot}008w}\text{, as before.}$$

When there is no load, $w=0$; then $l=51{\cdot}8+0=51{\cdot}8$;

$\therefore$ the natural or unstretched length of the spring is $51{\cdot}8$ cm.

Note. When we say that A, B are constants in the formula, we mean that their values remain the same for *different loads*. They do not, of course, remain the same for different springs. If we take a different spring, we shall still find that the stretched length and the load are connected by a relation of the form $l=B+A\,.\,w$, but there is no reason why A and B should have the same values as before. In fact, we have seen that B cm. represents the natural unstretched length of the spring; and the value of A depends on the elasticity of the spring, because the increase in the length is A cm. for each increase of 1 gram in the weight of the load.

Exercise IX. *f*.

[The examples of this exercise should be solved graphically and by calculation.]

1. If a spiral spring is supporting at its end a load of W lb., its length is $(a+bW)$ inches, where a, b are constants for the spring. When there is no load, the length is 15 in.; when the load is $2{\cdot}5$ lb., the length is $16{\cdot}5$ in.; find a, b. What load will make the stretched length $17{\cdot}1$ in.?

2. A temperature of t degrees Fahrenheit is the same as $(at+b)$ degrees Centigrade, where a, b are constants. The freezing point of water is 32° F. or 0° C., and its boiling point is 212° F. or 100° C. Find a, b. Hence express 0° F. in Centigrade and express 200° C. in Fahrenheit.

3. If a man's income is £x, where $160<x\leqslant400$, he pays £$(ax+b)$ tax. On incomes of £300, £360 the tax is £22 10s., £30 respectively. Find a, b. Find also the tax on an income of £200.

4. By using a machine, I can raise a load of W lb. by a pull of P lb. Find whether P and W appear to be connected by

the relation $P = aW + b$, where a, b are constants, if $P = 6$ when $W = 20$, and $P = 10$ when $W = 30$, and $P = 16$ when $W = 45$. If so, find a and b.

5. On a railway journey the charge for n lb. of luggage is $(an - b)$ pence where a, b are constants, so long as the charge is not negative. The charge for 70 lb. is 4d. and for 112 lb. is 1s. 4d.; find a, b. How much luggage is allowed free?

6. Marks which run from 26 to 78 are to be scaled so as to run from 0 to 100; an original mark x becomes a scaled mark $p + qx$ where p, q are constants. Find p and q. Find also the scaled mark corresponding to an original mark of 47.

7. After n years' service, a man's salary, £S a year, is given by $S = a + nb$, where a, b are constants. After 4 years it is £200 a year, and after 12 years it is £320 a year. Find a, b. Find also his salary for his first year and his 7th year.

8. Water is running out of a tank so that after n minutes the number of gallons, g, left in the tank is given by $g = a - bn$, where a, b are constants. The number of gallons left in the tank after 5 minutes and 11 minutes is 700 and 520 respectively. Find a, b. How much water was in the tank at the beginning?

9. If a polygon has n sides, the sum of its angles is $(a + bn)$ right angles, where a, b are constants. What is the sum of the angles of a figure with (i) 3 sides, (ii) 4 sides? Hence find a, b.

10. A framework is formed by rods jointed together at their ends. In order that the framework may be just rigid, the number of joints, j, and the number of rods, r, must be connected by the formula $r = a + bj$, where a, b are constants. If $j = 4$, then $r = 5$, and if $j = 6$, then $r = 9$. Can you see that this is true?

Now find the values of a, b. How many rods are required to make a framework just rigid if there are 10 joints?

The relation between r and j is the same as in the following statement: to make an accurate copy of a polygon with j corners, it is necessary to make r measurements. Why is this?

The remaining section of this chapter is added in order to complete the argument; it is not needed, however, in the chapters that follow, so that it may be reserved, without inconvenience, for a second reading.

Indeterminate Equations.

We have already seen that if two unknowns, x and y, are connected by a single relation, it is possible to find an unlimited number of pairs of values of x and y which satisfy the given condition.

Thus, if $2x+3y=32$, we can say that if $y=10$, $x=1$ or if $y=8$, $x=4$ or if $y=4$, $x=10$ or if $y=5$, $x=8\frac{1}{2}$ or if $y=-6$, $x=25$; etc.

There is a simple method of representing graphically pairs of values of x and y.

Coordinate Axes.

Take two perpendicular lines $X'OX$, $Y'OY$, called **coordinate axes**, intersecting at a point O, called the **origin**, and graduate the axes in any convenient manner. The line $X'OX$ is called the **x-axis** and the line $Y'OY$ is called the **y-axis**.

To represent the pair of values $x=2$, $y=4$, start from O and travel 2 units along $\overrightarrow{OX}$, x-wards, and then travel 4 units in the direction $\overrightarrow{OY}$, y-wards. We then arrive at the point A in Fig. 194. We say that the point A represents the *pair* of values $x=2$, $y=4$, and we call 2 the *x-coordinate* of A and 4 the *y-coordinate* of A, and we speak of A as the point (2, 4). The x-coordinate is always named first; if the coordinates of a point are (3, 5), then the x-coordinate is 3, and the y-coordinate is 5.

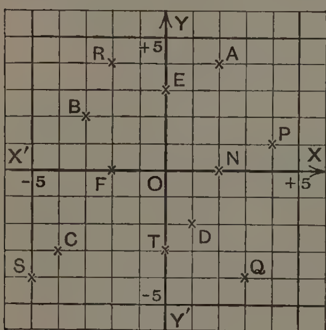

FIG. 194.

The coordinates of a point are directed numbers, (2, 4) is short for $(+2, +4)$. We choose $\overrightarrow{OX}$ and $\overrightarrow{OY}$ as the positive x-direction and the positive y-direction from O, then $\overrightarrow{OX'}$ and $\overrightarrow{OY'}$ are the negative directions from O. To represent the pair of values $x=-3$, $y=2$, start from O and travel (-3) units x-wards, *i.e.* $+3$ units along OX', then travel 2 units y-wards; we then arrive at the point B in Fig. 194; and we speak of B as the point $(-3, 2)$. Similarly in Fig. 194, C is the point $(-4, -3)$, D is the point $(1, -2)$, E is the point $(0, 3)$, F is the point $(-2, 0)$.

We therefore see that any pair of values of x and y can be represented by a point, and that any given point represents a definite pair of values of x and y.

If then we are given an equation connecting x and y, although it is impossible to find the numerical value of x and the numerical value of y, it is possible to find any number of pairs of values of x and y satisfying the equation and each of these pairs can be represented by a point: we therefore obtain a series of points, each of which represents one solution of the equation, and so the series of points may be said to represent graphically the equation.

Example VIII. Represent graphically $3x+2y=6$.

$$2y=6-3x\,; \quad \therefore\ y=\frac{6-3x}{2}.$$

We can therefore construct a table of values as follows:

$x=-4$	-3	-2	-1	0	1	2	3	4
$y=9$	7·5	6	4·5	3	1·5	0	$-1·5$	-3

Now take two coordinates axes and represent each pair of values of x and y by a point.

A glance at Fig. 195 shows that *all the points lie on a straight line.* This is not surprising, because the method employed is identical with that used for representing graphically the function $\frac{6-3x}{2}$. But further, *if we take any other point whatever on this straight line*, and measure its x-distance and its y-distance, *i.e.* its coordinates, the pair of values of x and y so obtained must satisfy the given equation $3x+2y=6$.

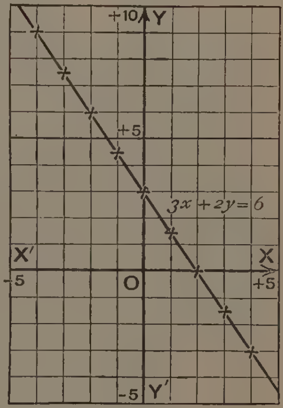

FIG. 195.

We therefore say that the equation $3x+2y=6$ is represented by the straight line in Fig. 195, and we call $3x+2y=6$ the **equation of the straight line.**

It will be found that every equation of the form $ax+by+c=0$, where a, b, c are constants, is represented by a *straight line*: but equations which involve x^2 or y^2 or xy or $\frac{1}{x}$, etc., are usually represented by curves.

We have seen on p. 188 that simultaneous linear equations in two variables can be solved graphically. The process which has been used is equivalent to drawing the straight lines which represent the given equations and noting the coordinates of the point of intersection.

In order to construct the straight line representing the given (first degree) equation, it is best to take *three* pairs of values of x and y; the straight line is fixed by *two* points, and the third point serves as a check.

In general, it is of course much quicker to solve two ordinary simultaneous equations by calculation than by drawing; moreover, the drawing method usually gives only approximate results. But the graphical method illustrates an important idea which has a large number of applications in more advanced work.

Exercise IX. *g*.

1. In Fig. 194, write down the coordinates of

(i) P; (ii) Q; (iii) R; (iv) S; (v) T; (vi) N.

2. Draw freehand two perpendicular lines as axes and graduate each of them by eye from -5 to $+5$; mark in the points $A(0, 3)$; $B(2, 0)$; $C(-4, 0)$; $D(0, -2)$; $E(1, 3)$; $F(-1, -2)$; $G(-2, 3)$; $H(3, -2)$.

3. Which of the following points lie on the line

$$5x+3y=52;$$

A, $(-1, 19)$; B, $(8, 4)$; C, $(4, 8)$; D, $(5, 9)$; E, $(10{\cdot}4, 0)$; F, $(14, -6)$; G, $(-3, 22)$?

4. Find a, b if the line $y=ax+b$ passes through the points $(0, 2)$; $(1, 5)$. Draw this line on squared paper; measure the coordinates of another point on this line and use them to check your values of a, b.

5. In Fig. 196, the equation of the line EF is $5y=3x+15$.

(i) What is OE? (ii) What is OF? (iii) If $OM=2$ units, what is MP? (iv) If $MP=6$ units, what is OM? (v) If $ON=-7$ units, what is NQ? (vi) If $NQ=-3$ units, what is ON?

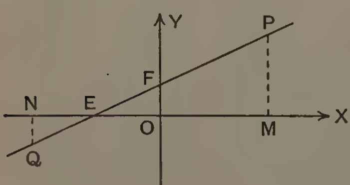

Fig. 196.

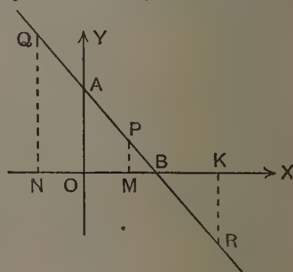

Fig. 197.

6. In Fig. 197, the equation of the line AB is $4x+3y=12$.

(i) What is OA? (ii) What is OB?

(iii) If $OM=2$ units, what is MP?

(iv) If $MP=1$ unit, what is OM?

(v) If $OK=5$ units, what is KR?

(vi) If $KR=-2$ units, what is OK?

(vii) If $ON=-2$ units, what is NQ?

(viii) If $NQ=5$ units, what is ON?

Solve graphically the following simultaneous equations; check by calculation.

7. $y=x+2,$ $x+y=5.$

8. $2y-x=4,$ $x+2y=3.$

9. $x=y,$ $5x+4y=20.$

10. $x+2y+5=0,$ $3y=2x+6.$

11. $3x=5y.$ $\frac{x}{2}+\frac{y}{3}=1.$

12. $\frac{x}{3}-\frac{y}{5}=1,$ $\frac{x}{4}+\frac{y}{3}+1=0.$

13. If $xy=48$, find the values of y corresponding to $x=16$, 12, 10, 8, 6, 4, 3, -3, -4, -6, -8, -10, -12, -16. Plot the results on squared paper and show a part of the curve whose equation is $xy=48$.

14. Draw the curve whose equation is $y=x^2-3x$ between $x=5$ and $x=-2$.

Draw with the same axes and scale the line whose equation is $y=\frac{1}{2}x$.

Hence solve graphically $y=x^2-3x$; $y=\frac{1}{2}x$.

15. Draw the curve whose equation is $10y=x^3$ from $x=-2$ to $x=3$: draw with the same axes and scale the line whose equation is $10y=4x+1$. Hence solve graphically the equation $x^3=4x+1$.

SUPPLEMENTARY EXERCISE. S. 6.

1. Solve $3x+5=8y+4=7x-7y-1$.

2. Solve $0=3x-7y-5=y-x+3$.

3. If $p=2q$ and $q+r=1$, what is the value of pq when $r=2$?

4. Solve $u+3v=2u+5v=2v-2$.

5. Solve $2(a+b+2)=3(2a-3b+3)=6(a-b)$.

6. If $2P+3Q=9$ and $3P+2Q=16$, find the value of $3P-2Q$.

7. Find two numbers x, y satisfying the relation $7x-2y=38$, if one of them is three times the other.

8. If $3r+4s=-119$ and $5r-11s=102$, prove that $r=s$.

9. If $a+b=9$ and $x=3a+7b$, express x in terms of a only.

10. Solve

$$3(7x-19)-2(11y-21)=4,\quad 5(7x-19)-3(11y-21)=7.$$

11. Solve $\dfrac{1}{x}+y=15,\ \dfrac{1}{x}-y=7.$

12. Solve $\dfrac{1}{x}+\dfrac{1}{y}=20,\ \dfrac{1}{x}-\dfrac{1}{y}=8.$

13. Solve $\dfrac{2}{u}+\dfrac{1}{v}=5,\ \dfrac{1}{u}+\dfrac{2}{v}=7.$

14. Solve $x^2+2y=10,\ 2x^2+7y=11.$

15. Solve $p^2+2q^2=17,\ p^2-q^2=5.$

16. Solve $2xy-3y=-14,\ 3xy-8y=-6.$

17. Solve $pq+13p=19,\ pq-11p=67.$

18. Solve $W(x-2)=10,\ W(x-1\frac{1}{2})=13\frac{1}{2}.$

19. If $x(1-y)=y(1-z)=1$, prove that $z(1-x)=1$.

20. If $x(a-b)=y^2$ and $y(a+b)=x^2$, express $(a+b)(a-b)$ in terms of x, y.

21. Solve $\dfrac{x+y}{x-y}=\dfrac{5}{3},\ \dfrac{x+y}{15}-\dfrac{x-y}{12}=\dfrac{1}{4}.$

22. Solve $\dfrac{x+y}{3}-\dfrac{x-y}{5}=1,\ \dfrac{x-y}{3}-\dfrac{x+y}{4}=2.$

23. Solve $\dfrac{x+1}{3}=\dfrac{2x+y+1}{2}+1,\ \dfrac{y-3}{4}=\dfrac{2x+y-2}{3}-1.$

24. Solve $x+2y=8,\ y+2z=13,\ z+2x=9.$

25. Solve $x+y+z=5,\ 2x-y+z=0,\ 4x+y+2z=6.$

26. If $x+3y+7z=14$ and $x+4y+10z=17$, find the *numerical* value of $x+y+z$.

27. Two angles of a triangle are $x°$, $y°$; two angles of another triangle are $(x-2y)$ degrees and $(y+70)$ degrees. The triangles are equiangular. Find x and y. [Three sets of answers.]

28. A number of two digits is equal to 7 times the sum of its digits; prove that the number formed by reversing the digits is equal to 4 times the sum of its digits.

29. A man can row u miles per hour in still water; to row upstream from A to B takes him three times as long as to row back from B to A; the stream runs at v miles per hour. Find v in terms of u.

30. A man's age in 1887 A.D. was equal to the sum of the digits of the year "$18xy$" A.D. in which he was born. Find one equation between x and y and hence find the year in which he was born.

31. If one side of a triangle is three times another side, prove that the ratio of the shortest side to the perimeter lies between $\frac{1}{6}$ and $\frac{1}{8}$.

32. There is a steady wind blowing from the North. An aeroplane A flying due North passes a stationary captive balloon B and after travelling 5 miles passes a balloon C drifting with the wind. After another 10 minutes, A turns back and arrives at B at the same moment as C does. If A flies at its maximum speed throughout, find the velocity of the wind.

33. The velocity of flow in a full pipe of diameter d inches, laid on a slope, is v ft. per sec., where $v=\dfrac{ad}{1+b\sqrt{d}}$, a and b being constants for a given fall. On a slope of 1 in 100, the velocity is 1·6 ft. per sec. for a diameter of 16 inches and is 2·9 ft. per sec. for a diameter of 3 feet. Find the values of a, b to two significant figures, for this slope. What should be the velocity for a pipe of diameter 25 inches, with the same slope?

CHAPTER X

COMMON-DIFFERENCE SERIES

Example I. A man starts with a salary of £250 a year and receives annual increases of £20 a year. Make a table of values for his salary in successive years and obtain a general formula.

The table of values is as follows :

Year of service -	1st	2nd	3rd	4th	5th	6th
Salary in £ - -	250	270	290	310	330	350

The set of numbers

$$250, 270, 290, 310, 330, \ldots$$

is called a **common-difference series**, and we say that the common difference is $+20$. Each number in the set is called a **term** ; 250 is the *first term*, 270 is the *second term*, and so on.

To find a general formula, we calculate the salary for the nth year of service : this is the nth term in the above set of numbers.

The 1st term is 250.

The 2nd term is $250+20$.

The 3rd term is $250+20+20=250+2\times 20$.

The 4th term is $250+20+20+20=250+3\times 20$,

and so on.

Thus the 10th term is $250+9\times 20$.

In general, the nth term is

$$250+(n-1)\times 20=250+20n-20=230+20n;$$

$\therefore$ for the nth year of service, the salary is £$(230+20n)$.

As we have previously seen, p. 206, the expression $230+20n$ is a linear function of n. The condition that the graph of a function of n is a straight line is that the function increases (or decreases) by equal amounts for equal increases in n. Consequently the nth term of any common-difference series is a linear function of n and is of the form $b+a\,.\,n$, where b, a are constants, *i.e.* b and a do not alter with n.

If x, y, z are three consecutive terms of a common-difference series, y is called the **Arithmetic Mean** between x and z.

We have $y-x=z-y$; $\therefore\ 2y=z+x$;

$$\therefore\ y=\frac{z+x}{2}.$$

Note. The Arithmetic mean between two numbers is the average of the two numbers.

If $x, y_1, y_2, y_3, y_4, \dots y_n, z$ are consecutive terms of a common-difference series, $y_1, y_2, y_3 \dots y_n$ are called the **n Arithmetic Means** between x and z.

Exercise X. *a*.

1. A man, after obtaining a certain post, saves £30 his first year, £36 his second year, £42 his third year, and continues to increase his annual saving by £6 each year. What does he save in his 12th year? Obtain a general formula for the amount he saves in his nth year of service.

2. A man one year after retirement has a bank-balance of £500 and each year after that this balance is reduced by £40. Make a table showing his bank-balance in successive years and obtain a general formula for his balance n years after retirement.

3. A swimming bath has a plane sloping floor; the depth of water is indicated by posts at equal distances down the bath. The reading on the first post is 12 ft., on the second post is 11 ft. 6 in. What is the reading on the fifth post? Obtain a general formula.

4. A marble rolls down an inclined groove; the distances it travels in successive seconds are 5 cm., 15 cm., 25 cm., 35 cm., etc. How far does it travel in the 8th second? Obtain a general formula.

5. With the data of No. 3, the reading on the last post in the bath is 4 ft. How many posts are there altogether?

6. The temperature of the water in a boiler is rising at a steady rate and is recorded in degrees Fahrenheit at equal intervals of time as follows: 82°, 88°, 94°, 100°, etc. The last reading taken was 190°. How many readings were taken? What was the nth reading?

Find the nth number in each of the following sets:

7. 1, 3, 5, 7, 9, ...

8. 40, 37, 34, 31, 28, ...

9. $2\frac{1}{2}, 3\frac{1}{4}, 4, 4\frac{3}{4}, 5\frac{1}{2}, \dots$.

10. $7, 4, 1, -2, -5, \dots$.

11. $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \dots$.

12. $\frac{3}{4}, \frac{5}{8}, \frac{7}{12}, \frac{9}{16}, \frac{11}{20}, \dots$.

Find the number of terms in the following common-difference series :

13. 4, 9, 14, ... , 64.

14. 8, 5, 2, ... , -25.

15. 6, 5·9, 5·8, ... , 3·5.

16. 22, 29, 36, ... , 169.

17. Find 3 numbers between 20 and 40, so that the five numbers you now have will form a common-difference series.

18. In the same way as in No. 17, insert five arithmetic means between 15 and 60.

19. Find the least number greater than 100 which belongs to the following common-difference series :

(i) 6, 13, 20, 27, ... ; (ii) 1·3, 1·6, 1·9, 2·2, ...

20. There are 8 terms in a common-difference series ; the first is 24 and the last is 80. What is the second ?

21. The nth term in a set of numbers is $5n-7$. Write down the first 5 terms. Does the set form a common-difference series ? If so, what is the common difference ?

22. The last term in a common-difference series of 16 terms is 100 ; the common difference is 3. What is the first term ?

23. The last term in a common-difference series of n terms is l ; the common difference is d. What is the first term ?

Sums of Common-Difference Series.

It is suggested that before the general method is taken, there should be some oral work of the following kind :

Example II. Fig. 198 shows a pattern of crosses arranged in a square, 5 in each side.

The total number is 5^2.

If we add them up according to the partitions, we obtain $1+3+5+7+9$.

$$\therefore\ 1+3+5+7+9=5^2;$$

$\therefore$ the sum of the first 5 odd numbers is 5^2.

Find by a similar plan the sum of (i) the first 8 odd numbers, (ii) the first n odd numbers. What is the nth odd number ?

In this way we can prove that the sum of n terms of the common-difference series 1, 3, 5, 7, 9, ... is n^2

By using this result,

$$1+3+5+7+9+\ldots+(2n-1)=n^2,$$

Can you say what the following sum is ?

$$2+4+6+8+10+\ldots+(2n).$$

Can you then find the value of the following set ?

$$1+2+3+4+5+\ldots+n.$$

Example III. Fig. 199 shows a pattern of crosses arranged in a square, 5 in each side as before. If we add them up along the diagonals, we obtain $1+2+3+4+5+4+3+2+1$.

$$\therefore\ 1+2+3+4+5+4+3+2+1=5^2;$$
$$\therefore\ 2(1+2+3+4+5)=5^2+5;$$
$$\therefore\ 1+2+3+4+5=\tfrac{1}{2}(5^2+5).$$

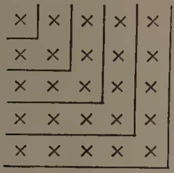

FIG. 198.

FIG. 199.

Find by a similar plan the sum of (i) the first 8 integers, (ii) the first n integers ?

Example IV. In Fig. 200, $\angle AOB=45°$; portions of length 1 inch are marked off from O along OA, and perpendiculars are drawn to OA to cut OB; then the lengths of these perpendiculars are 1, 2, 3, 4, 5, ... inches. Suppose there are seven portions.

The areas of the successive portions of the figure are $\frac{1}{2}$, $\frac{1}{2}(1+2)$, $\frac{1}{2}(2+3)$, $\frac{1}{2}(3+4)$, ... , $\frac{1}{2}(6+7)$ sq. in.

But the total area of the figure, a right-angled triangle, is $\frac{1}{2}\times 7\times 7$ sq. in.

$$\therefore\ 1+(1+2)+(2+3)+(3+4)+\ldots+(6+7)=7^2;$$
$$\therefore\ 2(1+2+3+\ldots+6)+7=7^2;$$
$$\therefore\ 2(1+2+3+\ldots+6+7)=7^2+7;$$
$$\therefore\ 1+2+3+\ldots+6+7=\tfrac{1}{2}(7^2+7).$$

What result is obtained by taking (i) 10 portions, (ii) n portions ?

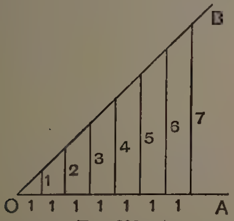

FIG. 200.

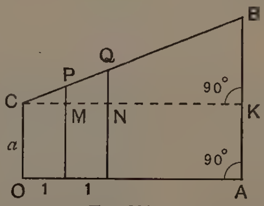

FIG. 201.

Example V. In Fig. 201, portions of length 1 inch are marked off along OA and perpendiculars are drawn to OA to cut CB; $OA=8$ inches; $OC=a$ inches, $PM=d$ inches, $QN=2d$ inches, etc.

Use the figure to show that

$$\tfrac{1}{2}(a+\overline{a+d})+\tfrac{1}{2}(\overline{a+d}+\overline{a+2d})+\ldots+\tfrac{1}{2}(\overline{a+7d}+\overline{a+8d})$$
$$=\tfrac{1}{2}\times 8\times(a+\overline{a+8d})$$

and deduce that

$$a+\overline{a+d}+\overline{a+2d}+\ldots+\overline{a+8d}=\tfrac{1}{2}\times 9(a+\overline{a+8d}).$$

Can you generalise this result ?

Example VI. Fig. 202 represents a number of equal rods, the lower portions of which are painted black and the upper portions white.

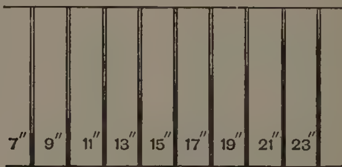

FIG. 202.

The lengths of the portions painted black are shown on the figure ; the smallest length is 7 inches and the lengths increase by equal amounts of 2 inches.

Since there are 9 rods in the figure, the length of the black portion of the last rod is $7+8\times 2=7+16=23$ inches.

Now suppose that each rod is $7+23=30$ inches long ; then the lengths of the portions painted white are, starting from the left, 23, 21, 19, ... , 9, 7 inches.

$\therefore$ the total length painted black is exactly the same as the total length painted white.

But the total length of all the rods is 9×30 inches.

$$\therefore \text{ the total length painted black} = \frac{9\times 30}{2} \text{ inches};$$

$$\therefore\ 7+9+11+13+15+17+19+21+23=9\times\frac{7+23}{2}.$$

Use a similar method to obtain the sum of the common-difference series :

$$5+8+11+14+17+20+23+26.$$

Example VII. What is the average of the following numbers : 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43 ? What is their sum ?

These numbers form a common-difference series. Reading forwards from the left, they increase by 4 ; reading backwards from the right, they decrease by 4.

$$\therefore\ 3+43=46=7+39=11+35=15+31=19+27;$$

$\therefore$ the average of each pair is $\frac{1}{2}$ of $46=23$; there is also a middle term, which is 23;

$$\therefore \text{ the average of the complete set is } 23.$$

But there are 11 terms in all;

$$\therefore \text{ their sum}=11\times 23.$$

We can also find the sum as follows:

Let $s=3+7+11+15+19+23+27+31+35+39+43.$

Then $s=43+39+35+31+27+23+19+15+11+7+3;$

$\therefore$ by addition,

$$2s=46+46+46+46+\ \ldots\ldots\ldots\ldots\ldots\ldots\ +46$$
$$=11\times 46;$$

$$\therefore\ s=\frac{11\times 46}{2}=11\times 23.$$

General Method.

Suppose the first term of a common-difference series is a and that the common difference is d, then the terms of the series are

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots.$$

$$\therefore \text{ the } n\text{th term is } a+(n-1)d.$$

Suppose there are n terms and that the last term is l.

$$\text{Then } \mathbf{l=a+(n-1)d.}$$

Let the sum of n terms be s.

If the last term is l, the last term but one is $l-d$, and the last term but two is $l-2d$, and so on.

Then $s=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l.$

Now write this expression for s backwards.

$$\therefore\ s=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a;$$

$\therefore$ adding

$$2s=(a+l)+(a+l)+(a+l)+\ldots+(a+l).$$

But there are n brackets.

$$\therefore\ 2s=n(a+l);$$

$$\therefore\ \mathbf{s=\frac{n}{2}(a+l).}$$

Further, $$l=a+(n-1)d;$$

$$\therefore\ s=\frac{n}{2}[a+a+(n-1)d];$$

$$\therefore\ \mathbf{s=\frac{n}{2}[2a+(n-1)d].}$$

A common-difference series is also called an **Arithmetical Progression** or, more shortly, an **A.P.**

Exercise X. *b*.

Find the sum of the following common-difference series :

1. $7+8+9+\ldots+100$.

2. $22+19+16+\ldots$; 20 terms.

3. $5\cdot3+7\cdot2+9\cdot1+\ldots$; 25 terms.

4. $11+13\frac{1}{2}+16+\ldots+36$.

5. $23+27+\ldots+67$.

6. $2+4+6+\ldots$; n terms.

7. $1+2+3+\ldots$; $2n$ terms.

8. $20+17+\ldots+(-25)$.

9. The sum of the common-difference series $3+\ldots+59$ is 465 ; find the number of terms and the common difference.

10. The sum of 12 terms of the common-difference series $4+\ldots$ is 246, find the last term and the common difference.

11. The sum of 20 terms of a common-difference series, whose last term is 121, is 1280 ; find the first term and the common difference.

12. A shop sells various sizes of tin kettles, and the prices of successive sizes rise by equal amounts. The smallest costs 2s. and the largest 8s. ; it costs £2 10s. to buy one of every kind. How many kinds are there ? What is the cost of the smallest but one ?

13. A Chinese nest of boxes is made so that the boxes are contained one within another ; the outermost weighs 2 oz. and the innermost $\frac{1}{8}$ oz., and their weights decrease by equal amounts. The whole nest weighs 17 oz. ; how many boxes are there in it ? If this nest is enclosed in one more box, what would the new weight be ?

14. A firm sends out a set of sample iron bolts, the difference in weight of successive sizes being the same. The smallest weighs $1\frac{1}{2}$ oz., the largest $10\frac{1}{2}$ oz., and the total weight of a complete set is 4 lb. 14 oz. How many bolts are there in the set ? What is the weight of the largest but one ?

15. Find the sum of all numbers less than 100 which are not divisible by 5.

16. What is the sum of all numbers divisible by 12 between 1000 and 5000 ?

17. A man, starting a business, loses £250 the first year, £170 the second year, £90 the third year ; if the same improvement continues, what is his total gain or loss after 12 years ?

18. *A* starts with a salary of £500 a year and receives a yearly increase of £100 a year ; *B* starts at £500 a year and receives a half-yearly increase of £50 per half-year. How much has each received altogether after 10 years ?

19. A square is divided into 25 equal squares, and the numbers 1, 2, 3, ... , 25, are arranged in these squares so that the sum of the numbers in each row and each column is the same. This arrangement is called a "magic square." What is the sum of the numbers in each row ?

20. In a potato race, ten potatoes are placed in a line with the starting point at intervals of two yards, and the nearest is 20 yards from the starting point, to which each potato must be brought separately. How many yards must a competitor run if he can reach one yard both picking up and putting down the potato ?

21. In a group of 12 people, every one writes once to every one else ; how many letters are written altogether ? [Count them as follows : how many letters does *A* write and receive ? How many letters does *B*, not counting correspondence with *A* ? How many letters does *C*, ignoring *A*, *B* ? And so on.]

22. A gramophone record has a spiral line cut on it winding from the outer edge, which is a circle of radius 6 inches, to the inner edge, which is a circle of radius 2 inches. The record revolves 80 times a minute and takes 4 minutes to play. Assuming that the spiral path is equivalent to a number of equally spaced concentric circles, find the total length of the path in yards to the nearest yard.

23. Postal orders are issued for all multiples of sixpence from 6d. to 21s. inclusive, except for 20s. 6d. ; the commissions are as follows : 6d. to 2s. 6d., one penny ; 3s. to 15s., three halfpence ; 15s. 6d. to 21s., two pence. (i) Find the money received by cashing a complete set. (ii) Find the cost of purchasing a complete set.

24. *A* starts with a salary of £500 a year and receives a yearly increase of £20 a year ; *B* starts at the same time at

£300 a year and receives a yearly increase of £50 a year. After how many years will the total sum received by B exceed that received by A?

25. Find the value of $\dfrac{101+103+105+\ldots+199}{1+3+5+\ldots+99}$.

26. A clerk's commencing salary is £100 a year; he is offered a choice between a yearly rise of £5 and a rise of £22 every 4 years. Calculate the total sum he will receive in the course of 33 years under each arrangement.

27. If you write down the set of numbers 4, 11, 18, 25, ..., will one of the numbers be 690 or not?

28. If a, b are positive integers and if $a<b$, find the sum of all integers from a to b inclusive.

29. If the first two terms of an A.P. are a, b, what is the nth term?

30. The first and last terms of an A.P. are a, l; there are n terms in all. What is the second term?

PART III

The new work of Part III. should be supplemented by occasional revision of the ground covered in Parts I.-II. The following papers, A. 21-30 and B. 11-20, are inserted here to facilitate such revision.

It is also suggested that the subject matter of Chapter XV. should be taken concurrently with the work of Chapters XI.-XIV.

EASY REVISION PAPERS. A. 21-30

A. 21

1. A penny weighs $\frac{1}{3}$ oz., a halfpenny weighs $\frac{1}{5}$ oz. What is the weight of (i) x shillings' worth of pennies, (ii) one shilling's worth of coppers of which n are halfpennies ?

2. (i) Multiply $(a-b-c)$ by $-b$ and $(b-c-a)$ by $-c$, and $(c-a-b)$ by $-a$; add the results.

(ii) Divide $a^2bc+ab^2c-abc^2$ by $-abc$.

3. Solve : (i) $x(x-3)+5=x(x+1)-2$;

(ii) $5x-12y=1$, $x+y=7$.

4. The duty on a bottle of wine is trebled, and this increases the cost of a bottle from 1s. 9d. to 2s. 2d. If the increase in the cost of a bottle is the same as the increase in the duty, find how much the duty was at first.

5. If the sum of two numbers is equal to twice their difference, prove that their product is equal to three times the square of the smaller number.

A. 22

1. The temperature of the air, in degrees Fahrenheit, taken every two hours, is recorded as follows :

12	2 a.m.	4	6	8	10	12	2 p.m.	4	6	8	10
53	51·5	51	55	61	67	71	73	73	69	60	56

Draw a temperature graph and find the time (to the nearest hour) at which it is (i) hottest, (ii) coldest.

2. Solve: (i) $\frac{2x-7}{2}+x=\frac{1}{2}$;

(ii) $\frac{3-x}{11-x}=\frac{1}{16}$.

3. A parcel is just small enough to go by post if $l+c=6$, where l is its length and c its girth, in feet. What is the greatest girth for a parcel $2\frac{1}{2}$ ft. long? What is the greatest length for a parcel, the end of which is 9 in. square?

4. If three-fifths of a number exceeds one-quarter of it by 63 what is the number?

5. The energy of a moving body, weighing W lb., whose speed is v ft. per sec., is $\frac{Wv^2}{64}$ foot-pounds. One body A weighing 56 lb. has a speed of 20 ft. per sec. and another body B weighing 35 lb. has a speed of 32 ft. per sec. Compare the energies of A and B.

A. 23

1. A stone moves $(5t+2t^2)$ ft. in t seconds, for all values of t. How far will it go in (i) 3 sec., (ii) 4 sec., (iii) the fourth second?

2. Simplify: (i) $(x^2-2x)\div x$; (ii) $(2x-x^2)\div(-x)$;

(iii) $\frac{x^2-2x}{x}-\frac{2x-x^2}{-x}$; (iv) $\frac{y-z}{x}\div\frac{z-y}{-x}$.

3. Simplify $p^2-\{p(2p-3)-7\}$.

4. Solve: (i) $0{\cdot}2x+1{\cdot}5=2{\cdot}2$;

(ii) $x=2y,\ 7x-9y=55$.

5. A boy buys 15 dozen papers at 8d. a dozen. He sells some of them at a penny each and sends back the rest, getting 7d. a dozen for them. If he gained 3s. $6\frac{1}{2}$d., how many did he sell?

A. 24

1. (i) If a ship has unloaded $\frac{a}{b}$ of its cargo, how much remains? Can a and b have any values [*e.g.* $a=\frac{1}{2}$, $b=\frac{3}{4}$; $a=5$, $b=4$]?

(ii) The census years are 1801, 1811, 1821, ... 1911, 1921. Give a formula for these years.

2. Solve: (i) $\frac{x}{3}+\frac{x}{5}=2$; (ii) $\frac{2}{W}=\frac{3}{5}$; (iii) $\frac{y}{3}=-\frac{1}{5}$.

3. A boat is sailing at v miles an hour through the water; there is a current of u miles an hour flowing against her. How fast is she advancing? What does this become if $v=4$ and $u=6$, and what does it mean?

4. (i) Subtract d from $a+(n-1)d$.

(ii) Add $(a+d)$ to $a+(n-2)d$.

5. The telegraph posts at the side of a railway are 60 yd. apart; a train passes x posts each minute; show that its speed is about $2x$ miles per hour.

A. 25

1. The height of the water at a post is $(+16)$ ft. at high water and (-10) ft. at low water. What is the height, halfway between high and low water? What is the fall of the tide?

2. Is $\left(\frac{217}{344}\right)^2$ greater or less than $\frac{217}{344}$?
For what value of x (positive or negative) is x^2 less than x?

3. (i) Subtract $x-(y+z)$ from $(x+y)-z$.

(ii) Subtract $3-5x+x^2$ from $2x^2-x-2$.

4. The hour hand of a clock is half the length of the minute hand, and its tip moves through 2 in. every hour; how many feet does the tip of the minute hand move through in t hours.

5. If the triangle in Fig. 203 is isosceles, prove that it is equilateral.
Find x and y, the angles being measured in degrees.

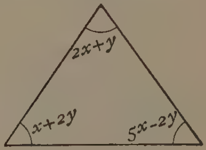

FIG. 203.

A. 26

1. If $a=-1$, $b=2$, find the values of

(i) a^2; (ii) $2a-b$; (iii) $\frac{a}{b}-\frac{b}{a}$; (iv) a^3-b^3.

Simplify $\frac{1}{3}(2x-3y)-\frac{1}{2}(x-y)$.

2. If x guineas is the same as £$(x+1)$ ys., find y in terms of x.

3. Solve: $\frac{x}{200-x}=\frac{3}{5}$.

4. Explain why you cannot solve:

(i) $6x-8y=5,\ 9x-12y=7\frac{1}{2}$;

(ii) $6x-8y=5,\ 9x-12y=8$.

5. I am thinking of two numbers; if I halve the first it is one more than the second; if I add 4 to the second it is one-third of the first. What are the numbers?

A. 27

1. If 1 inch $=p$ cm. and 1 metre $=q$ inches, find a relation between p and q.

2. (i) Find the H.C.F. of $16abc^2$ and $24a^2bc$;

(ii) Add $\frac{1}{16abc^2}$ to $\frac{1}{24a^2bc}$; (iii) simplify $\frac{16abc^2}{24a^2bc}$.

3. When it is x o'clock at Greenwich, it is $\left(x+\frac{l}{15}\right)$ o'clock by local time in longitude l degrees east. What is the time (i) in Bombay (long. 74° E.), (ii) in New York (long. 74° W.), when it is 6 p.m. at Greenwich?

What does the formula give for the local time in Bombay, when it is 10 p.m. at Greenwich and for the local time in New York, when it is 2 p.m. at Greenwich? Explain the answers.

4. Solve: (i) $\frac{8x}{15}-\frac{7x}{6}=\frac{5}{2}$;

(ii) $\frac{8x}{3}+2y-5=x+\frac{y}{2}+8=\frac{6x}{5}+3y+\frac{9}{5}$.

5. If $a+b=6$, $b+c=7$, $c+a=8$, find a, b, c.

A. 28

1. The charges for parcels by train from London to Hastings are as follows:

Weight in lb.	2	3	4	5	6	7	8	9	10
Charge in pence	4	5	6	7	8	9	10	11	12

Find a formula for the charge on a parcel weighing W lb. Where W is an integer and $2 < W < 10$.

2. Multiply $2x^2+xy-y^2$ by $-x$ and subtract from the result $-y(x^2+xy-2y^2)$.

3. The real weight of a body is A gm.; the error made in weighing it is not more than p times its true weight. Between what two values does the weight recorded by the machine lie? What are the answers if $p=\frac{1}{100}$?

4. Solve (i) $\frac{1}{2}(x+5)-\frac{2}{3}(x-2)=\frac{7}{2}$;

(ii) $y=2x+5$, $y=x-7$.

5. The perimeter of a semicircular area is 8 inches; find the diameter correct to $\frac{1}{10}$ inch. [Take $\pi=\frac{22}{7}$.]

A. 29

1. The expression $ax+b$ is equal to 2·9 when $x=7$ and is equal to 7·1 when $x=14$. If a, b are constants, find their values.

2. Coal costs £P per ton. At this rate, how many pence does 14 lb. cost? If the selling price is 25 per cent. more than this, for what is 14 lb. sold?

3. In an experiment in mechanics, the formula $a=\frac{w-w_1}{W+w}\cdot g$ is used. Find w_1 if $W=900$, $w=100$, $g=980$, $a=70$.

4. Simplify: (i) $\frac{8x^3-6x^2}{-2x}$; (ii) $\frac{ab}{a^3b+a^2b^2+ab^3}$.

5. In a division in the House of Commons, 378 members voted and the Government had a majority of 62. How many voted for the Government?

A. 30

1. A ship starts from O and for 2 hours steams due E. at 8 miles an hour through the water: she then turns and steams at the same rate due West. There is a steady current flowing East at 3 miles an hour. How far *East* of O is the ship t hours after leaving O, if $t>2$? Substitute 7 for t in your answer and interpret the result.

2. A cask containing 6 gallons of cider weighs 78 lb. The same cask containing $3\frac{1}{2}$ gallons weighs 53 lb. Find the weight of the cask.

3. Solve: (i) $0{\cdot}25x-0{\cdot}4(x-3)=1{\cdot}02$;

(ii) $3x-10y=-19,\ y=\frac{x-2}{5}$.

4. A quantity of an alloy of copper and tin weighs 112 lb. Another alloy is made which contains $\frac{1}{4}$ less copper and $\frac{1}{3}$ less tin and weighs 82 lb. What fraction by weight of the first alloy was tin?

5. The top and bottom marks in an examination are 125 and 34; these are scaled so as to run from 100 to 50. An original mark x becomes $ax+b$. Find a and b. What was the mark which when scaled became 90?

HARDER REVISION PAPERS. B. 11-20

B. 11

1. Which day of the week is the $(7n+4)$th day after a Sunday, if n is a positive integer?

What can you say about x and y, if the xth day of April and the yth day of May are both Tuesdays?

2. Fill in the brackets in the following:

(i) $2x-3\equiv2(\qquad)+1$;

(ii) $3x+5\equiv x(\qquad)$.

3. If $a=-2$, $b=-1$, $c=0$, find the value of:

(i) $(a-b)-(a-c)$, (ii) $(a-b)^2-(a-c)^2$.

4. Solve: $xy+3x=7,\ 2x-xy=1$.

5. The Food Controller in Feb. 1917 allowed per week to each person 4 lb. of bread; instead of any portion of the bread, three-quarters of its weight of flour might be used. If a person's allowance weighs $3\frac{2}{3}$ lb., how much bread does he have?

B. 12

1. Subtract $\frac{3}{4x^2}$ from $\frac{4}{5x}$. For what positive values of x is the second expression less than the first?

2. The product $P \cdot v$ is constant. If $P = 1\frac{1}{2}$ when $v = 2\frac{1}{2}$, what is the value of v when $P = 3$? What is the value of P when $v^2 = \frac{1}{4}$?

3. If a stone is thrown up with velocity u ft. per second, its height after t seconds is $(ut - 16t^2)$ feet. If $u = 80$, find its height after (i) 5 sec., (ii) 6 sec. Explain the answers.

4. With the formula of No. 3, find the velocity with which the stone is thrown up, if it is at the same height after 6 sec. as after 4 sec.

5. ABC is a triangle such that $AB = AC = 12$ in. and $\angle BAC = 90°$; P, Q are points on AB, AC such that $AP = AQ = x$ in.; $PAQN$ is a square; PN, QN are produced to cut BC at H, K; prove that the area of $PNKB$ is $\frac{3x}{2}(8 - x)$ sq. in. and find graphically the value of x for which this area is greatest and what the maximum area is.

B. 13

1. If a ship is L ft. long and B ft. broad, its tonnage is reckoned as $\frac{(L - \frac{3}{5}B)B^2}{188}$. What is the tonnage of a ship 100 ft. long, 24 ft. broad, to the nearest whole number? What does the formula become if $L = 10B$?

2. Find the H.C.F. of $12a^2b^2$ and $18\,abc$.
What is the H.C.F. of $12a^2b^2$ and $18\,abc$, if $a = 1$, $b = 2$, $c = 2$?

3. Solve: (i) $\frac{x}{3} - \frac{x-2}{4} = 1$; (ii) $3x = 7y$, $11x - 9y = 20$.

4. A train arrives at x o'clock and is y hours late. If it had been $(y + 1)$ hours late, when would it have arrived? At what time was it due? Assume that $y < x < 11$.

5. The following table gives the normal weight of men of various heights:

Height in ins.	62	63	64	65	66	67	68	69	70	71	72
Weight in lb.	126	133	139	142	145	148	155	162	169	174	178

Represent the relation graphically.
By drawing another graph, find roughly the greatest error in saying that a man of height x inches should normally weigh $(5x - 182)$ lb.

B. 14

1. A gardener draws a roller w ft. wide at v miles an hour. How many minutes will he take to roll half an acre of grass ?

2. Express in words, $x \times \frac{9}{5} \equiv 2x - \frac{1}{10}$ of $(2x)$. Use this rule to write down the value of $73 \times \frac{9}{5}$.

3. Add $a^2 - a - 1$ and $1 - a - a^2$; subtract $1 - a - a^2$ from $a^2 - a - 1$. Multiply the two results together.

4. A man would pay the same income tax whether he paid 1s. 6d. in the £ on all his income above £130 or 2s. in the £ on all his income above £160. What is his income ?

5. Solve: $\quad W(x-5)=9, \quad W(2x-1)=45.$

B. 15

1. Solve: $$\frac{1-2x}{5}=\frac{3x+1}{4}-\frac{5x-1}{3}.$$

2. If $2A+3B=21$ and $7A-4B=1$, prove that $3A+5B=34$.

3. (i) Simplify $xy\left(\frac{2}{x^2}-\frac{3}{y^2}\right)-2\left(\frac{x}{y}+\frac{y}{x}\right)$;

(ii) If $y=\dfrac{x^2+2x}{x+3}$, find the value of x^2y when $x=2$.

4. The formula for the area of a trapezium can be written in the form $\frac{1}{2}h(a+b)$. Express this in terms of h if $2b=3a=4\frac{1}{2}h$.

5. A man has 22 minutes to get to the station, two miles away. If he takes a motor bus which travels a mile in 8 minutes, at what distance from the station can he get out and walk, if he walks a mile in 16 minutes ?

B. 16

1. If $x=-1$, $y=-3$, find the values of

$$\text{(i) } (1-x)^2; \quad \text{(ii) } xy^2; \quad \text{(iii) } \frac{y-1}{x-1}.$$

2. Solve: $$\frac{x}{0{\cdot}5}-\frac{x+1}{0{\cdot}8}=1.$$

3. If $y=ax+b$ where a, b are constants, and if $x=4$ when $y=1$, and $x=7$ when $y=7$, find the values of a, b; find also the value of x when $y=0$. Sketch roughly the graph of $ax+b$ for the above data.

4. An oak beam supported at its ends will just carry at the middle a load of $\dfrac{3bd^2}{2l}$ cwt., where b, d are the breadth, depth in inches and l is the length in feet. What is the greatest safe load at the middle point, if the beam is 12 ft. long, 1 ft. broad, 18 in. deep ?

5. Three loops of a string are made once round a box in the three different directions; their lengths are 7 in., 9 in., 12 in.; find the volume of the box.

B. 17

1. (i) Subtract $1-(a-b)$ from $a-(1-b)$;

(ii) Divide $(-6ab^2)^3$ by $3a^2b^2$.

2. (i) What number exceeds 23 by the same amount that 23 exceeds 17?

(ii) What number exceeds a by the same amount that a exceeds b?

3. (i) Find a value of x that makes $4(x-11)$ equal to $x-1$.

(ii) What values of x, y make the expressions $x+y$, $5(2x-y)$, $x+2(y-1)$ equal?

4. A chain is composed of n circular links, each of which is d inches in external diameter and is made of material $\frac{1}{2}$ inch thick. What is the total length of the chain, when taut?

5. A heavy uniform bar AB, of length $2a$ inches, is supported at its ends (see Fig. 204). C is the mid-point of AB and P is any

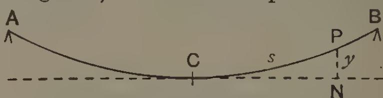

FIG. 204.

point on AB, s inches from C. The height, $PN=y$ inches, of P above C is given by $y=\dfrac{s^2(6a^2-s^2)}{100a^3}$. Find (i) the height of B above C, (ii) the height of the mid-point of BC above C.

B. 18

1. Solve: $$\frac{1}{3x}-\frac{1}{4x}=\frac{1}{60}.$$

2. Simplify (i) $2x(x-3x^2)-x^2(1-2x)-(-2x)^2$;

(ii) $1+\dfrac{4x-2y}{2x}-\dfrac{9y-x}{3y}$.

3. (i) Multiply the reciprocal of $\dfrac{1}{x^2}$ by $2x$.

(ii) What are the square roots of $12xy^2 \times 3x^3$?

4. If a vessel is L ft. long, B ft. broad, her tonnage is T, where $T=\dfrac{\left(L-\dfrac{3B}{5}\right)B^2}{188}$. Make L the subject of this formula.

5. In Fig. 205, HQ, QP, PK are arcs of circles with C, A, B

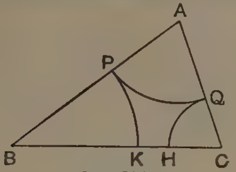

FIG. 205.

as centres; $BC = a$ in., $CA = b$ in., $AB = c$ in.; calculate the length of AP, (i) if $HK = \frac{1}{2}BC$, (ii) if H coincides with K.

B. 19

1. (i) Subtract $a - 1$ from $1 - a$.

(ii) Multiply the sum of $3x^2 - 2x + 1$ and $x^2 + x + 2$ by the excess of $2(x-1)$ over $x - 2$.

2. Simplify (i) $(ab)^2 - (-ab)^2$; (ii) $\frac{x-1}{-1\frac{1}{2}} + \frac{2x-1}{4\frac{1}{2}}$.

3. Solve: $$\frac{x+y}{3} = \frac{2x}{21} = \frac{x-y-2}{15}.$$

4. (i) If $3a + 2b - c = 7$, what is the value of b when $a = 2$, $c = -1$?

(ii) If $5x + 2y = 47$, what is the least integral value of x which makes y a negative integer?

5. In Fig. 206, $TH = 5$ in., $HK = 3$ in., $KT = 6$ in.; calculate the length of HN.

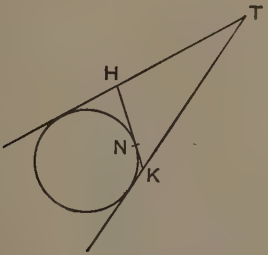

FIG. 206.

B. 20

1. One sort of candle burns for $2t$ hours and another sort for $3t$ hours; how many candles of the first kind will last as long as c candles of the second kind, using only one at a time?

2. Simplify (i) $x-\frac{x^2}{2x}$; (ii) $\sqrt{\left(a^2+\frac{9a^2}{16}\right)}$; (iii) $(3x)^3\div(-2x)^2$.

3. If $3x+4y=1$, $2x-5y=16$, $ax-4y=2$, find a.

4. ABC is a triangle such that $AB=4x$ in., $BC=7x$ in., $CA=5x$ in.; D is a point on BC, such that $BD=\frac{1}{2}DC$; find how much longer it is from A to D *via* C than *via* B.

5. One end A of a light rod of length l inches is built into a wall and a weight is suspended from the other end (see Fig. 207). The

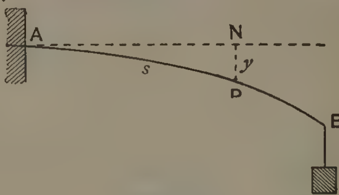

FIG. 207.

consequent deflection at any point P, s inches from A, is y inches, where $y=\frac{s^2(3l-s)}{100l^2}$. If C is the mid-point of AB, and if D, E are the mid-points of AC, CB, find the deflections at B, C, D, E.

CHAPTER XI

FACTORS AND PRODUCTS.

Single Term Factors.

CONSIDER the rectangles in Fig. 208.

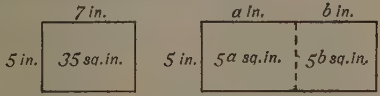

FIG. 208.

The area of the first is $5 \times 7 = 35$ sq. in.; 5 and 7 are factors of 35; $\frac{35}{5} = 7$; $\frac{35}{7} = 5$.

The area of the second is $(5a + 5b)$ sq. in. $= 5(a + b)$ sq. in.; 5 and $(a + b)$ are factors of $(5a + 5b)$; $\dfrac{5a + 5b}{5} = a + b$; $\dfrac{5a + 5b}{a + b} = 5$.

EXERCISE XI. *a*.

Draw figures to illustrate areas representing the following expressions:

1. $3(p + q)$.
2. $4(x + 2)$.
3. $a(l + m)$.
4. $2p + 2q$.
5. $15a + 15b$.
6. $pa + pb$.
7. $3x + 3y + 3z$.
8. $x^2 + xy$.
9. $ax + bx + cx$.

Draw figures to illustrate the following statements:

10. $\dfrac{3x + 3y}{3} = x + y$.
11. $\dfrac{4p + 4q}{p + q} = 4$.
12. $\dfrac{ab + ac}{a} = b + c$.
13. $\dfrac{ab + ac}{b + c} = a$.
14. $\dfrac{lx + ly + lz}{l} = x + y + z$.
15. $\dfrac{a^2 + ab}{a + b} = a$.

Short Division.

If there is a factor common to each term of an expression, we can put the expression into factors by short division.

Example I. Factorise $6a^2 - 10ax + 2a$.

$2a$ is a factor of each term of the expression $6a^2 - 10ax + 2a$, therefore it is a factor of the whole expression.

$$\begin{array}{r|l} 2a & 6a^2 - 10ax + 2a \\ \hline & 3a - 5x + 1 \end{array}$$

$$\therefore\ 6a^2 - 10ax + 2a = 2a(3a - 5x + 1).$$

Exercise XI. *b*.

1. (i) What is the meaning of $10(x+3y)$? What is the result if the brackets are removed?

(ii) Factorise $10x + 30y$.

2. What is the quotient if (i) $7x + 7y$ is divided by 7; (ii) $7x + 7y$ is divided by $x + y$; (iii) $ax + ay$ is divided by a?

3. Factorise (i) $ax - 3ay$; (ii) $4a^6 - 2a^3x^2$.

4. What is the quotient if (i) $cx + c$ is divided by c; (ii) $cx + c$ is divided by $x + 1$?

5. Factorise (i) $ay - a$; (ii) $x^2 - x$.

6. Divide $2x^3 - 6x^2y$ by $2x^2$.

7. Factorise (i) $3a^4 - 15a^2b^2$; (ii) $a^6 + a^2$.

8. (i) Factorise $xy + xz$; (ii) Divide $xy + xz$ by $y + z$.

9. (i) Factorise $3b^2c - 6bc^2$; (ii) Divide $3b^2c - 6bc^2$ by $3bc$; (iii) Divide $3b^2c - 6bc^2$ by $b - 2c$; (iv) Divide $3b^2c - 6bc^2$ by $c(b - 2c)$.

10. Is x^2 a factor of (i) $x^4 + 2xy^2$; (ii) $x^5 + x^3$?

Factorise the expressions, Nos. 11-21:

11. $2a^3 - 6a^2b$.
12. $10 - 15x^2$.
13. $xy^2 - x^2y$.
14. $x^4 + x^2 + x$.
15. $c^3 + c^2d + cd^2$.
16. $3a^2b^2 - 6abc + 15a^2b$.
17. $x^2(x+y) + x^6$.
18. $a^2(a-b) + 3a^2b$.
19. $5a^2 - a^{10}$.
20. $7x^4y^3 - 35x^3y^5 - 14x^5y^4$.
21. $15a^3b^3 - 12a^2b^4 - 18ab^5$.

Find the following, Nos. 22-31.

22. H.C.F. of $2xy$ and $3x^5$.

23. H.C.F. of $2a+2b$ and $3a+3b$.

24. L.C.M. of $2a+2b$ and $3a+3b$.

25. H.C.F. of $a(x+y)$ and $a(x-y)$.

26. L.C.M. of $a(x+y)$ and $a(x-y)$.

27. L.C.M. of x^2+x and $xy+y$.

28. L.C.M. of $6x^3-2x^2y$, $18x^2y-6xy^2$.

29. H.C.F. of x^3+x, x^2+1.

30. H.C.F. of a^3+a^2b, a^2b+ab^2, a^2c+abc.

31. L.C.M. of $3a^2-3ab$, $2ab-2b^2$, $5ab$.

32. Simplify (i) $\frac{3x}{4x}$; (ii) $\frac{3(x+y)}{4(x+y)}$; (iii) $\frac{5a-5b}{6a-6b}$; (iv) $\frac{4a+4b}{6a-6b}$.

33. Simplify (i) $\frac{2x}{x(x+y)}$; (ii) $\frac{x}{x(x+y)}$; (iii) $\frac{a(b+2c)}{b+2c}$.

34. Fill in the blank spaces in

(i) $\frac{3}{10}=\frac{}{50}$; (ii) $\frac{a}{b}=\frac{}{bc}$; (iii) $\frac{a}{b}=\frac{}{b(x+y)}$; (iv) $b=\frac{}{a+b}$.

35. Express $\frac{1}{x+y}$ and $\frac{1}{x}$, so that each has $x(x+y)$ as its denominator.

36. Express $\frac{a}{a-b}$ and $\frac{a}{b}$, so that the two fractions have equal denominators.

37. What is the L.C.M. of x and $x+y$?

Simplify $$\frac{1}{x}+\frac{1}{x+y}.$$

38. What is the L.C.M. of $a+b$ and $a-b$?

Simplify $$\frac{1}{a+b}+\frac{1}{a-b}.$$

39. Find the H.C.F. and L.C.M. of $x(x-y)$ and $(x-y)^2$.

Simplify (i) $\frac{(x-y)^2}{x(x-y)}$; (ii) $\frac{1}{x(x-y)}-\frac{1}{(x-y)^2}$.

40. Fill in the blank spaces in $x = \frac{}{xy}$ and $a = \frac{}{a+b}$.

Simplify the following, Nos. 41-52 :

41. $x + \frac{xy}{x-y}$.

42. $\frac{1}{x} - \frac{y}{x^2+xy}$.

43. $a - \frac{ab}{a+b}$.

44. $\frac{1}{x^2} - \frac{1}{x^2+x}$.

45. $\frac{1}{2} - \frac{x}{3x-3y}$.

46. $\frac{1}{a} - \frac{a}{a^2-b^2}$.

47. $\frac{1}{2a-2b} - \frac{1}{3a-3b}$.

48. $\frac{1}{y} + \frac{x}{y(x+y)} - \frac{2}{x+y}$.

49. $\frac{1}{a} + \frac{1}{b} - \frac{1}{a+b}$.

50. $2 + \frac{x}{x+y} - \frac{2y}{x}$.

51. $\frac{a}{a^2+ab} - \frac{b}{(a+b)^2}$.

52. $a + b - \frac{a^2}{a-b}$.

Solve the following equations, 53-60 :

53. $\frac{1}{x+1} + \frac{2}{5} = \frac{1}{2}$.

54. $\frac{3}{2x+8} - \frac{1}{x+4} = \frac{1}{8}$.

55. $\frac{2x-1}{4} - \frac{3}{x} = \frac{x}{2}$.

56. $1 - \frac{x-2}{x+2} = \frac{3}{x}$.

57. $\frac{x}{x-1} - 1 = 1 - \frac{x-2}{x}$.

58. $\frac{1}{x} + \frac{x}{x+1} = 1 - \frac{1}{x}$.

59. $\frac{2}{x} + \frac{3}{x+5} = \frac{20}{x(x+5)}$.

60. $\frac{x-1}{x-3} + \frac{2}{x} = 1$.

EXTRA PRACTICE EXERCISES. E.P. 9.

Fractions and Equations.

Simplify the fractions and solve the equations in the following examples :

1. $\frac{1}{2} - \frac{1}{x-2}$.

2. $\frac{1}{2} - \frac{1}{x-2} = \frac{1}{4}$.

3. $\frac{1}{x} + \frac{1}{2x} - \frac{1}{3x}$.

4. $\frac{1}{x} + \frac{1}{2x} - \frac{1}{3x} = \frac{1}{9}$.

5. $\frac{x}{3} - \frac{x+1}{5}$.

6. $\frac{x}{3} - \frac{x+1}{5} = 1$.

7. $1-\frac{1}{x+1}$.

8. $2+\frac{1}{2x+1}=1$.

9. $\frac{1}{3x}-\frac{1}{3x+3}$.

10. $\frac{1}{3x}-\frac{1}{3x+3}=\frac{1}{4x}$.

11. $x+3-\frac{x^2}{x-3}$.

12. $\frac{x}{x+2}-\frac{1}{x}=1$.

13. $\frac{x}{x-1}-\frac{1}{x}-1$.

14. $\frac{x}{2x-4}-\frac{x}{3x-6}$.

15. $x+4-\frac{x^2}{x-4}=1$

16. $\frac{x}{x-1}+\frac{1}{x}=1$.

17. $\frac{x}{x+1}+\frac{3x}{2+2x}-2$.

18. $\frac{1}{x}-\frac{x+1}{x(x-1)}+\frac{1}{x-1}$.

19. $\frac{5}{x-2}-\frac{3}{x}=\frac{12}{x(x-2)}$.

20. $\frac{x}{2x-4}-\frac{x}{3x-6}=1$.

Multiplication and Division.

Note. Examples II. and III. below and most of Exercise XI. c., which illustrate the methods of long multiplication and division, may be omitted at a first reading, without affecting the work that follows. But the method of Example IV. should be explained and practised, see Exercise XI. c., Nos. 1-9, 16-20, and Exercise XI. f.

Example II. Multiply $2x-5$ by $3x-4$.

$$\begin{array}{rrr} 2x & -5 & \\ & 3x & -4 \\ \hline 6x^2 & -15x & \\ & -8x & +20 \\ \hline 6x^2 & -23x & +20 \end{array}$$

Answer.

Example III. Divide $4x^2-6xy+7y^2$ by $2x-y$.

$$\begin{array}{r|l} & \quad 2x - 2y \\ \hline 2x-y & 4x^2-6xy+7y^2 \\ & 4x^2-2xy \\ \cline{2-2} & \qquad -4xy+7y^2 \\ & \qquad -4xy+2y^2 \\ \cline{2-2} & \qquad\qquad 5y^2 \end{array}$$

Quotient $2x-2y$; Remainder $5y^2$.

Note. When multiplying two expressions together or when dividing one expression by another, *arrange both expressions* either *in ascending powers* or *in descending powers* of some common letter. In the above examples, the expressions are arranged in descending powers of x.

Example IV. Show that $x^2 - y^2 = (x+y)(x-y)$.

$$\begin{aligned}(x+y)(x-y) &= x(x-y)+y(x-y)\\ &= x^2 - xy + yx - y^2\\ &= \mathbf{x^2 - y^2}.\end{aligned}$$

Note. The expression $\mathbf{x^2 - y^2}$ is called **the difference of two squares.** Its factors must be committed to memory.

Exercise XI. c.

Work out the following, Nos. 1-15 :

1. $(x+3)(x+4)$.
2. $(3x+2)(2x+5)$.
3. $(x+2)(x-7)$.
4. $(2x+3)(x-2)$.
5. $(3x-1)(2x-1)$.
6. $(2x-3y)(x+5y)$.
7. $(a+2b)(a-2b)$.
8. $(5x-4)^2$.
9. $(2a-5b)^2$.
10. $(x^2-3x+5)(x+2)$.
11. $(3x^2+x-7)(x-3)$.
12. $(a^2-ab+b^2)(a+b)$.
13. $(a^2+ab+b^2)(a-b)$.
14. $(x^2-2x+1)(x+2)$.
15. $(x^2+2x+4)(5x-3)$.
16. Multiply $x+a$ by $x+b$ and show that the answer can be written $x^2+x(a+b)+ab$.
17. Multiply $x+a$ by $x-b$ and group the x terms together as in No. 16.
18. Multiply $x-a$ by $x+b$ and group as in No. 16.
19. Multiply $x-a$ and $x-b$ and group as in No. 16.
20. Multiply $y-z$ by $y-1$; group the terms in y together.

Work out the following, Nos. 21-35 :

21. $(x^2+5x+4)\div(x+4)$.
22. $(6x^2+5x+1)\div(2x+1)$.
23. $(x^2+x-20)\div(x+5)$.
24. $(2x^2+5x-18)\div(x-2)$.
25. $(x^2+7x+10)\div(x+2)$ and $(x^2+7x+10)\div(x+5)$.
26. $(2x^2+9x+7)\div(2x+7)$ and $(2x^2+9x+7)\div(x+1)$.
27. $(x^2-3x-4)\div(x-4)$; factorise x^2-3x-4.
28. $(x^2+4)\div(x+2)$. Is $x+2$ a factor of x^2+4 ?
29. $(x^3+9x^2+22x+16)\div(x+2)$.
30. $(x^3-4x^2-5)\div(x-3)$.
31. $(18x^3-5x-2)\div(3x-2)$.

32. $(x^2+ax+bx+ab)\div(x+a)$; factorise $x^2+ax+bx+ab$.
33. $[x^2+(a+b)x+ab]\div(x+b)$. 34. $(x^2+xy+xz+yz)\div(x+y)$.
35. $(x^2-xy-xz+yz)\div(x-y)$; factorise $x^2-xy-xz+zy$.
36. One factor of a^2-b^2 is $a-b$. What is the other ?
37. One factor of $a^2+10ab+24b^2$ is $a+4b$. What is the other ?
38. One factor of $4x^2-44x+121$ is $2x-11$. What is the other ? Write down the two square roots of $4x^2-44x+121$.
39. One factor of a^3-b^3 is $a-b$. What is the other ?
40. One factor of a^3+b^3 is $a+b$. What is the other ?

Relation between a − b and b − a.

It is evident that $7+5=5+7$, and $12+3=3+12$, and $19+31=31+19$, etc.

All such statements are expressed by the formula

$$\mathbf{a+b=b+a}.$$

Further $7-5=2$ and $5-7=-2$; $\therefore\ (7-5)=-(5-7)$.
Also $12-3=9$ and $3-12=-9$; $\therefore\ (12-3)=-(3-12)$.
Also $19-31=-12$ and $31-19=12$; $\therefore\ (19-31)=-(31-19)$.
All such statements are expressed by the formula

$$\mathbf{a-b=-(b-a)}.$$

We can see that this relation is true by removing the bracket on the right-hand side.

Consequently, $$\frac{x}{a+b}=\frac{x}{b+a}.$$

But, just as $$\frac{2}{-3}=\frac{-2}{3}=-\frac{2}{3},$$

so $$\frac{x}{a-b}=\frac{x}{-(b-a)}=\frac{-x}{b-a}=-\frac{x}{b-a}.$$

Example V. Find the L.C.M. of $y+x$; $2(x-y)$; $3(y-x)$; y^2-x^2.

First arrange all the expressions in descending powers of x (or, if preferred, all in ascending powers of x).

They then become

$$x+y;\ 2(x-y);\ -3(x-y);\ -(x^2-y^2)$$

or $$x+y;\ 2(x-y);\ -3(x-y);\ -(x+y)(x-y).$$

$\therefore$ the L.C.M. is $\mathbf{6(x+y)(x-y)}$ or $\mathbf{6(x^2-y^2)}$.

Note. We could also say that the L.C.M. is

$$6(y^2-x^2) \quad\text{or}\quad -6(x^2-y^2).$$

Example VI. Simplify : $\dfrac{1}{x-y}+\dfrac{x}{y^2-x^2}$.

$$y^2-x^2=-(x^2-y^2)=-(x+y)(x-y).$$

$$\therefore \text{ the expression} = \frac{1}{x-y}+\frac{x}{-(x+y)(x-y)}$$
$$=\frac{1}{x-y}-\frac{x}{(x+y)(x-y)}$$
$$=\frac{(x+y)-x}{(x+y)(x-y)}=\frac{\mathbf{y}}{(\mathbf{x}+\mathbf{y})(\mathbf{x}-\mathbf{y})}.$$

Exercise XI. *d*.

1. If $x+y=10$ and $x-y=1$, write down the values of

(i) $y+x$; (ii) $y-x$; (iii) $\dfrac{1}{y+x}$; (iv) $\dfrac{1}{y-x}$; (v) $\dfrac{1}{y^2-x^2}$.

2. If $x-y=3$, write down, where possible, the *numerical* values of (i) $y-x$; (ii) $y+x$; (iii) x^2-y^2; (iv) $\dfrac{1}{2y-2x}$.

3. If $x-y=a$, write down, where possible, in terms of a, the values cf (i) $y+x$; (ii) $y-x$; (iii) $\dfrac{1}{y^2-x^2}$; (iv) $\dfrac{1}{3y-3x}$.

4. If $a-b=5$, find the values of

(i) $\dfrac{2}{a-b}-\dfrac{1}{a-b}$; (ii) $\dfrac{2}{a-b}+\dfrac{1}{b-a}$.

5. If $p-q=a$ and $x-y=b$, write down, where possible, in terms of a and b, the values of

(i) $\dfrac{p-q}{x-y}$; (ii) $\dfrac{p-q}{y-x}$; (iii) $\dfrac{q-p}{x-y}$; (iv) $\dfrac{q-p}{y-x}$;
(v) $(p-q)(y+x)$; (vi) $(p-q)(x-y)$; (vii) $(q-p)(y-x)$;
(viii) $(x-y)(q+p)$; (ix) $(q-p)^2$; (x) $(x+q)-(p+y)$.

6. Divide $a-b$ by $b-a$. 7. Divide $(a-b)^2$ by $b-a$.

8. Simplify $(2x-2y)\div(3y-3x)$.

9. Find the H.C.F. of $2(x-y)$ and $3(y-x)$.

10. Find the L.C.M. of a^2-ab and b^2-ab.

11. Find the L.C.M. of $a+b$, $a-b$, $b+a$, $b-a$.

Simplify the following, Nos. 12-25 :

12. $(x-y)(y-x)$.

13. $-\dfrac{(a+b)(a-b)}{(b+a)(b-a)}$.

14. $(a-b)^3 \div (b-a)^2$.

15. $\dfrac{2}{x-1}+\dfrac{1}{1-x}$.

16. $\dfrac{x}{x-y}+\dfrac{y}{y-x}$.

17. $\dfrac{1}{x^2-1}+\dfrac{1}{1-x^2}$.

18. $\dfrac{1}{(x-1)^2}-\dfrac{x}{(1-x)^2}$.

19. $1+\dfrac{a}{a-b}-\dfrac{b}{b-a}$.

20. $\dfrac{a}{ab-b^2}-\dfrac{b}{ab-a^2}$.

21. $\dfrac{1}{ab+a^2}+\dfrac{1}{ab+b^2}$.

22. $\dfrac{1}{x^2-x}+\dfrac{1}{1-x}$.

23. $\dfrac{2x}{x-1}-\dfrac{x}{1-x}-3$.

24. $\dfrac{2y}{x-y}+\dfrac{2y^2}{(y-x)(y+x)}$.

25. $\dfrac{x^2-xy}{y-x}-\dfrac{y^2-2xy}{2x-y}$.

Applications to Areas.

(1) Draw a rectangle $(a+b)$ inches long, $(c+d)$ inches wide; divide it into four rectangles, as shown in Fig. 209.

The area of the original rectangle is $(a+b)(c+d)$ sq. inches.

The areas of the four compartments are ac, ad, bc, bd sq. inches.

$$\therefore\ (a+b)(c+d)=ac+ad+bc+bd.$$

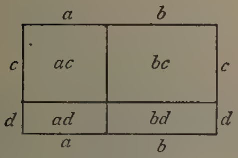

FIG. 209.

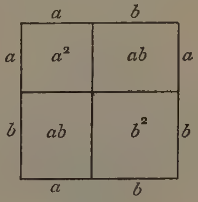

FIG. 210.

(2) Draw a square with side $(a+b)$ inches long; divide it into four rectangles, as shown in Fig. 210; two of these rectangles are squares.

The area of the original square is $(a+b)^2$ sq. inches.

The areas of the four compartments are a^2, ab, ab, b^2 sq. inches.

$$\therefore\ (a+b)^2=a^2+2ab+b^2.$$

(3) Draw a square with side a inches long; cut away from it a square of side b inches long, as shown in Fig. 211; the remainder has an area $(a^2 - b^2)$ sq. inches.

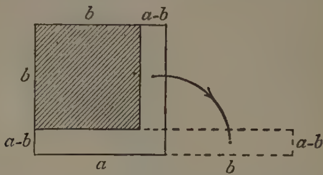

FIG. 211.

Now divide this remainder into two rectangles as shown and place them in line. They form a single rectangle of length $(a+b)$ inches and breadth $(a-b)$ inches; $\therefore$ the area of this remainder is $(a+b)(a-b)$ sq. inches.

$$\therefore\ \mathbf{a^2 - b^2 = (a+b)(a-b).}$$

(4) Fig. 212 represents a right-angled triangle, with its hypotenuse c inches long, and its other sides a, b inches long.

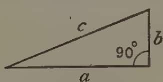

FIG. 212.

Draw two squares each with side $(a+b)$ inches long; take 4 copies of the given triangle and arrange them in the two different ways shown in figures 213, 214.

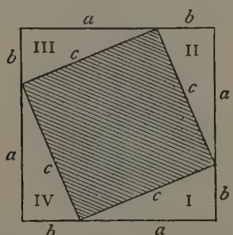

FIG. 213.

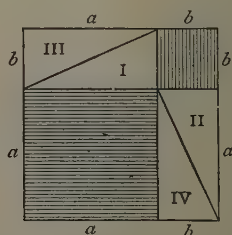

FIG. 214.

It can be proved by simple geometry that each shaded area in Figs. 213, 214 is a square, since the given triangle is *right-angled*.

But the shaded area in Fig. 213 differs from the area of the whole square by the same amount (viz. four times the area of the

given triangle), that the sum of the shaded areas in Fig. 214 differs from the area of the whole square.

But the shaded area in Fig. 213 is c^2 sq. inches, and the sum of the shaded area in Fig. 214 is (a^2+b^2) sq. inches.

$\therefore$ if the sides and hypotenuse of a right-angled triangle are a in., b in., and c in.,

$$\mathbf{a^2+b^2=c^2}. \qquad \textbf{Pythagoras' Theorem.}$$

We may also argue as follows :

The area of the given right-angled triangle is $\frac{1}{2}ab$ sq. inches.

$$\therefore \text{ the shaded area in Fig. } 213=[(a+b)^2-4\times\tfrac{1}{2}ab] \text{ sq. inches}$$
$$=a^2+2ab+b^2-2ab=a^2+b^2 \text{ sq. inches.}$$
$$\therefore\ c^2=a^2+b^2.$$

The Rectangle contained by Two Lines.

If AB and CD are any two given straight lines, the phrase " the rectangle contained by AB and CD," which is usually written simply as $AB \,.\, CD$, means the area of a rectangle such that one of its sides is equal to AB and the adjacent side is equal to CD.

Thus, if $AB=x$ in. and $CD=y$ in., the phrase " the rectangle contained by AB and CD " means the area of a rectangle x in. by y in. (see Fig. 215), which is xy sq. inches : and we should write

$$AB \,.\, CD=xy \text{ sq. inches.}$$

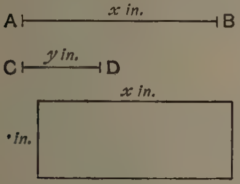

FIG. 215.

FIG. 216.

For example, the relation (2) above may be stated as follows :

If a line AB is divided at any point P (see Fig. 216), the square on AB is equal to the sum of the squares on AP and PB, together with twice the rectangle contained by AP and PB, and it may be written as follows :

$$AB^2=AP^2+PB^2+2AP \,.\, PB.$$

Here we are not using a letter to represent a number but to represent a point, and we mean by AB the number of units of length from A to B.

The context always shows when letters are being used in this sense.

Exercise XI. e.

1. Draw a rough figure to illustrate that
$$k(a+b+c)=ka+kb+kc.$$

2. Illustrate by a rough figure $(2a)^2=4a^2$.

3. Illustrate by a rough figure that $k(a-b)=ka-kb$.

4. Illustrate by a rough figure that
$$(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab.$$

5. Use Fig. 217 to show that $(a-b)^2=a^2-2ab+b^2$.

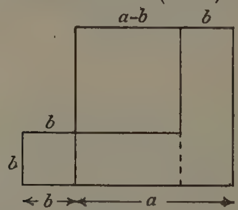

FIG. 217.

6. Illustrate by a rough figure that
$$(x+2)(x+3)=x^2+5x+6.$$

7. Illustrate by a rough figure that
$$(a-b)(c-d)=ac-ad-bc+bd.$$

8. In Fig. 218, A, B, C, D are any four points in order on

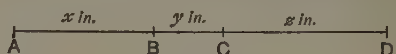

FIG. 218.

a straight line; prove that $AC \,.\, BD=AB \,.\, CD+AD \,.\, BC$. [Do not draw the rectangles.]

9. In Fig. 219, O is the mid-point of a line AB and P is any

FIG. 219.

point on AB; prove that $AP^2+PB^2=2AO^2+2OP^2$. [Do not draw the squares.]

10. Prove the result in No. 9, if P lies on AB produced.

11. AD is trisected at B, C; prove that $AD^2=AB^2+2BD^2$.

12. In Fig. 219, prove that $AB^2+AP^2=2AB \,.\, AP+PB^2$.

13. A straight line AB is bisected at C and produced to P, prove that $AP^2=4AC \,.\, CP+BP^2$.

14. $ABCD$ is a straight line; if $AB = CD$, prove that

$$AD^2 + BC^2 = 2AB^2 + 2BD^2.$$

15. The diagonals of the quadrilateral $ABCD$ in Fig. 220, cut at right angles at O; prove that the area of $ABCD$ equals $\frac{1}{2}AC \cdot BD$.

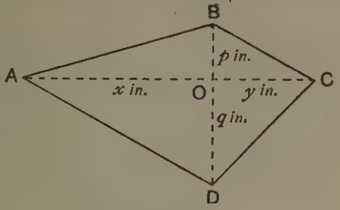

FIG. 220.

16. Prove that if the sides of a triangle are (x^2+1), (x^2-1), $2x$ inches long, they satisfy the condition that the triangle is right-angled. What is the result if $x = 2$?

17. In Fig. 221, prove that $b^2 - c^2 = x^2 - y^2$.

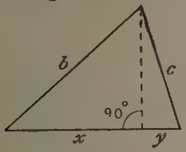

FIG. 221.

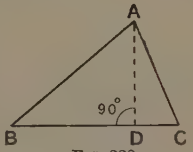

FIG. 222.

18. In Fig. 222, prove that $AB^2 = BC^2 + AC^2 - 2BC \cdot CD$. [Let $AB = x$ in., $BC = y$ in., $CA = z$ in., $CD = p$ in., $AD = h$ in.]

19. In Fig. 223, prove that $AB^2 = BC^2 + AC^2 + 2BC \cdot CD$. [Use method of No. 18.]

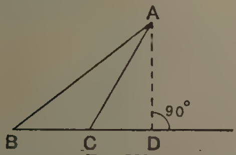

FIG. 223.

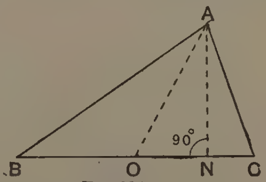

FIG. 224.

20. In Fig. 224, $BO = OC$; prove that

$$AB^2 + AC^2 = 2AO^2 + 2OB^2.$$

[Let $BO = OC = p$ in., $ON = q$ in., $AN = h$ in.]

21. In Fig. 225, $\angle BAC = 90°$; show that (i) $p^2 + q^2 = (x+y)^2$; (ii) $h^2 = xy$; (iii) $BA^2 = BD \cdot BC$.

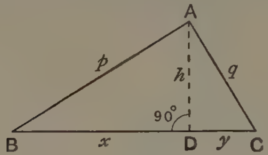

FIG. 225.

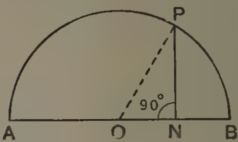

FIG. 226.

22. In Fig. 226, O is the centre of the semicircle;

$$OA = OP = OB = r \text{ in.}, \quad ON = x \text{ in.},$$

prove that $PN^2 = (r^2 - x^2)$ sq. in. $= AN \cdot NB$.

23. With the data and figure of No. 20, prove that

$$AB^2 - AC^2 = 2BO \cdot ON.$$

24. In Fig. 227, O is the centre of the semicircle; ii TP is a tangent, it is known that $\angle TPO = 90°$; prove that

$$TP^2 = TA \cdot TB.$$

[Let $AO = OP = OB = r$ in., $OT = x$ in.]

25. Show that the area between two concentric circles of radii a, b inches is $\pi(a+b)(a-b)$ sq. inches.

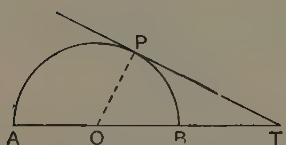

FIG. 227.

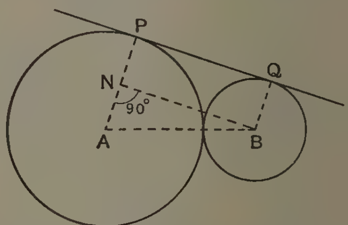

FIG. 228.

26. Show that if a box is x ft. high, $(x+1)$ ft. wide and (x^2+x) ft. long, a diagonal of the box is (x^2+x+1) ft. long.

27. Two circles, centres A, B, radii a in., b in., touch each other, as shown in Fig. 228; PQ is their common tangent. It is known that $\angle APQ = 90° = \angle BQP$; prove that

$$PQ = NB = 2\sqrt{(ab)}.$$

Multiplication by Inspection.

The product $(a+b)(x+y)$ equals $a(x+y)+b(x+y)$ equals
$$ax+ay+bx+by.$$

In expanding the product, we multiply each term in the one bracket by each term in the other. With a little practice, the product can be written down without any intermediate work.

Thus $(a-b)(x+y)=ax+ay-bx-by.$

Again $(x+3)(x+7)=x^2+7x+3x+21=x^2+10x+21,$

and $(x-3)(x+7)=x^2+7x-3x-21=x^2+4x-21.$

Here also, the intermediate steps need not be written down.

Exercise XI. *f*.

Write down the following products :

1. $(a+b)(c+d)$. 2. $(a+b)(c-d)$. 3. $(a-b)(c+d)$.

4. $(a-b)(c-d)$. 5. $(a+b)(a-b)$. 6. $(a+b)(a+b)$.

7. $(a-b)(a-b)$. 8. $(x+y)(x+z)$. 9. $(y+z)(y-z)$.

10. $(x+2y)(a-3b)$. 11. $(3x+2y)(2a-b)$. 12. $(x-y)(x-1)$.

13. $2(x-a)(y-b)$. 14. $(x+1)(x+2)$. 15. $(x+2)(x+3)$.

16. $(x+5)(x+7)$. 17. $(x-6)(x+9)$. 18. $(x-3)(x-8)$.

19. $(a+1)(a-11)$. 20. $(a-7)(a+3)$. 21. $(a-7)(a+7)$.

22. $(1+x)(2+x)$. 23. $(2-x)(7+x)$. 24. $(z-20)(z+2)$.

25. $(3+b)(8-b)$. 26. $(1+3y)(1+2y)$. 27. $(9-x)(4-x)$.

28. $(x+2y)(x-3y)$. 29. $(x-3y)(x-4y)$. 30. $(a-5b)(a+b)$.

31. $(2x+1)(x+4)$. 32. $(3x+2)(4x+5)$. 33. $(3x-2)(4x-5)$.

34. $(3x+2)(4x-5)$. 35. $(2x-7)(3x-1)$. 36. $(x+2y)(2x+y)$.

37. $(4x+y)(x-3y)$. 38. $(4+5x)(3+2x)$. 39. $(1-x)(7-8x)$.

40. $(3a+2b)(4b-a)$. 41. $(5x+11)(2x-7)$. 42. $(3-5x)(7-3x)$.

43. $(5-a)(5+a)$. 44. $(2x+1)^2$. 45. $(3y-1)^2$.

46. $(3b-2)^2$. 47. $(3y+2)(3y-2)$. 48. $(a+3b)^2$.

49. $(x+5y)(x-5y)$. 50. $(x-5y)^2$. 51. $(3a-4b)(3a+4b)$.

Factors by Grouping Terms.

If we are putting $ax+bx$ into factors, we say "x is a common factor, it goes into ax a times and it goes into bx b times."

$$x \,\underline{|\ ax+bx\ }$$
$$\underline{a+b}$$
$$\therefore\ \mathbf{ax+bx=x(a+b).}$$

Exactly in the same way, if we are putting

$$a(y+z)+b(y+z)$$

into factors, we say $(y+z)$ is a common factor

$$(y+z)\,\underline{|\ a(y+z)+b(y+z)\ }$$
$$\underline{a \qquad +b.}$$
$$\therefore\ \mathbf{a(y+z)+b(y+z)=(y+z)(a+b).}$$

In all factorisation, *if there is a common factor*, whether consisting of a single term or several terms [cf. x and $(y+z)$ above], always **take it out first of all and write it down first**, because, if you are using the division method, this is the order in which you ought to be thinking of the work. The divisor comes first, the quotient is found last.

Example VII. Factorise $a^2-ab-ac+bc$.

$$a^2-ab-ac+bc=(a^2-ab)-(ac-bc)$$
$$=a(a-b)-c(a-b)$$

But $(a-b)$ is a common factor; divide throughout by $(a-b)$; the quotient is $a-c$.

$$\therefore\ a^2-ab-ac+bc=\mathbf{(a-b)(a-c).}$$

Example VIII. Factorise $ab-ax-xy+by$.

$$ab-ax-xy+by=(ab-ax)-(xy-by)$$
$$=a(b-x)-y(x-b)$$
$$=a(b-x)+y(-x+b).$$

But $(b-x)$ is a common factor.

$$\therefore\ ab-ax-xy+by=\mathbf{(b-x)(a+y).}$$

Note. When you have factorised an expression, multiply the factors together in your head and so make sure that they give the expression with which you started.

EXERCISE XI. g.

[*Note.* In this exercise, Nos. 1-21 are intended for oral discussion.]

1. Is $x+1$ a factor of $2x+2$?
2. Is $x+1$ a factor of $2(1+x)$?
3. Is $1-x$ a factor of $x-1$?
4. Is $a-3$ a factor of $a+3$?
5. Is $x-y$ a factor of $3(y-x)$?
6. Is $x+y$ a factor of (i) $4(x-y)$; (ii) $ax+ay$; (iii) $-x+y$?
7. Is $x+y$ a factor of $a(x+y)+b(y+x)$?
8. Is $x+y$ a factor of $a(x+y)+b(x-y)$?
9. Is $a+b$ a factor of $a(x+y)+b(x+z)$?
10. What are the factors of $x(p+q)+y(p+q)$?
11. What are the factors of $x(p-q)+y(p-q)$?

Have the following expressions factors ? If so, state what they are, and verify by multiplication. *If there are no factors, say so.*

12. $a(x+y)+b(x-y)$.
13. $c(x+y)+d(y+x)$.
14. $x(x+y)+y(x+z)$.
15. $a(b+c)+b(a+c)$.
16. $x(a-b)+y(b-a)$.
17. $x(y+1)+y(x+1)$.
18. $x(2a+2)+y(a+1)$.
19. $a(2x+y)+2b(x+y)$.
20. $x(a+b)+a+b$.
21. $a(b-c)-b+c$.

Factorise the following expressions, *when possible ?* If there are no factors, say so.

22. $x^2+xy+xz+yz$.
23. $x^2-xy+xz-yz$,
24. $x^2-xy-xz+yz$.
25. $x^2-xy-xz-yz$.
26. $ap+aq+bp+bq$.
27. $ca-cd-bd+ba$.
28. $ax+ap-cx-pc$.
29. $a^2+ab+2a+2b$.
30. $a(x+y+z)+b(x+y+z)$.
31. $x^4+a^2x^2+b^2x^2+a^2b^2$.
32. $5cx+5dy-5cy-5dx$.
33. $x^6+x^4+x^2+1$.
34. $a^2-ab+a-b$.
35. $x^2-xy-x-y$.

36. $a^2 + ab - a - b$.

37. $x^2(x + a) - bc(x + a)$.

38. $a(x + y) + b(x - y)$.

39. $a(x + y) + c(y - x)$.

40. $2a + 6b + 3by + ay$.

41. $6ab - 3bx + 2ay - xy$.

42. $4x^2 - 2xy - 6xz + 3yz$.

43. $x^3 + x + x^2 + 1$.

44. $x^2 + (c + d)x + cd$.

45. $1 + a(x + x^2) + x$.

46. $(a + 2b)(x + y) + (a + 2c)(x + y)$.

47. $a(bx + cx) + bd + cd$.

48. $ax - bx + bq - aq$.

49. $z^4 + z^3 + z^2 + z$.

50. $2px + qy - py - 2qx$.

51. $ab - 12xy + 3bx - 4ay$.

52. $ax - 3 + a - 3x$.

53. $ac + ad - a^2 - cd$.

54. $x^2 - x + y - xy$.

55. $x^4 - x^3 + bx^2 - bx$.

56. $4 - 4x + cx - c$.

57. $l(a - b) + m(a + b)$.

58. $l(a - b) + m(b - a)$.

59. $2a^4 - 2a^3x + x - a$.

60. $1 + c^2 + cd + c^3d$.

61. $ax - 2ay + 2bx - by$.

CHAPTER XII

QUADRATIC FUNCTIONS

THE expression ax^2+bx+c, where a, b, c are constants, is called a **quadratic function of x**; a is the **coefficient of $\mathbf{x}^2$**, b is the **coefficient of x**, c is the **constant term** (see p. 44).

In order to factorise this expression, we shall use the method of "grouping."

Note. The case, where the coefficient of x^2 is unity, is taken first, for the sake of simplicity; but the method used is fundamentally equivalent to that followed in the general case. Those teachers, however, who prefer to *start* with the general case will find the treatment on p. 258; Exercises XII. *c.* and XII. *a.* will then be taken concurrently.

Factors of $\mathbf{x^2+px+q}$.

First consider the reverse process:

$$(x+3)(x+11)=x(x+11)+3(x+11)$$
$$=x^2+11x+3x+3\times 11=\mathbf{x^2+14x+33}.$$

If then we are given $x^2+14x+33$ and are asked to factorise it, we must break up $14x$ into two terms such that their product is $x^2\times 33$ or $33x^2$.

Now $\quad 33x^2=11x\times 3x$ and $14x=11x+3x$.

We therefore replace $14x$ by $11x+3x$.

$$\therefore\ x^2+14x+33=x^2+11x+3x+33$$
$$=x(x+11)+3(x+11)$$
$$=(\mathbf{x+11})(\mathbf{x+3}).$$

Example I. Factorise $x^2+10x+24$.

Replace $+10x$ by two terms whose product is

$$(x^2)(24)=24x^2=(4x)(6x).$$

$$\therefore\ x^2+10x+24=x^2+4x+6x+24$$
$$=x(x+4)+6(x+4)$$
$$=(\mathbf{x+4})(\mathbf{x+6}).$$

Check the answer by multiplication

$$(x+4)(x+6)=x^2+4x+6x+24=x^2+10x+24.$$

Example II. Factorise $x^2 - 3x - 70$.

Replace $-3x$ by two terms whose product is $x^2(-70)$

$$= -70x^2 = (-10x)(7x).$$

$$\therefore\ x^2 - 3x - 70 = x^2 - 10x + 7x - 70$$
$$= x(x-10) + 7(x-10)$$
$$= (x-10)(x+7).$$

Example III. Factorise $70 + 3x - x^2$.

Do not turn the expression round.

Replace $+3x$ by two terms whose product is

$$70(-x^2) = -70x^2 = (10x)(-7x).$$

$$\therefore\ 70 + 3x - x^2 = 70 + 10x - 7x - x^2$$
$$= 10(7+x) - x(7+x)$$
$$= (7+x)(10-x).$$

Example IV. Factorise $24a^2 + 2ab - b^2$.

Replace $+2ab$ by two terms whose product is

$$(24a^2)(-b^2) = -24a^2b^2 = (6ab)(-4ab).$$

$$\therefore\ 24a^2 + 2ab - b^2 = 24a^2 + 6ab - 4ab - b^2$$
$$= 6a(4a+b) - b(4a+b)$$
$$= (4a+b)(6a-b).$$

Note. Verify your factors by multiplying out either in full or by inspection.

Exercise XII. *a*.

Write down the following products, Nos. 1-9 :

1. $(x+3)(x+8)$. 2. $(x-3)(x+8)$. 3. $(x+3)(x-8)$.
4. $(x-3)(x-8)$. 5. $(1+x)(11+x)$. 6. $(1+x)(11-x)$.
7. $(x-y)(11x-y)$. 8. $(a-b)(11a+b)$. 9. $(p+3q)(p-5q)$.

Write down the coefficients of x in Nos. 10-15 :

10. $(x+30)(x+9)$. 11. $(x+50)(x-80)$. 12. $(x+8)(x-8)$.
13. $(x-12)(x-4)$. 14. $(x+7)(x+7)$. 15. $(5+x)(x-8)$.

Write down the constant term in Nos. 16-21 :

16. $(x-15)(x+10)$. 17. $(a-6)(a-8)$. 18. $(5-y)(y+3)$.
19. $(q+7)(q-4)$. 20. $(1-3x)(1+5x)$. 21. $(2+z)(z-2)$.

Factorise Nos. 22-74, check your answers by multiplication.

22. $x^2+(3+7)x+3\times 7$.
23. $x^2-(c+d)x+cd$.
24. x^2+6x+8.
25. $x^2+9x+20$.
26. $x^2-8x+15$.
27. $x^2-9x+20$.
28. x^2+3x+2.
29. x^2+7x+6.
30. $x^2-11x+10$.
31. $x^2-2x-35$.
32. $x^2-3x-70$.
33. $x^2+7x+12$.
34. $x^2+12x+36$.
35. x^2-8x+7.
36. x^2-4x+4.
37. $x^2+105x+500$.
38. x^2+2x+1.
39. $x^2+22x+40$.
40. $x^2-7x-44$.
41. $a^2+5a-14$.
42. a^2-25.
43. $z^2-9z-36$.
44. $l^2-50l+600$.
45. $a^2+7a-44$.
46. $y^2+6y-16$.
47. $z^2-13z+36$.
48. $b^2+20b+75$.
49. $x^2-8x-48$.
50. $t^2-10t-96$.
51. $k^2+k-132$.
52. $c^2-14c+45$.
53. $p^2+9p-52$.
54. $x^2+30x+200$.
55. $b^2-12b-108$.
56. $y^2-20y+96$.
57. $1+x-12x^2$.
58. $1-10a+21a^2$.
59. $1-2b-24b^2$.
60. $1+3x-10x^2$.
61. $10+7y+y^2$.
62. $24-5x-x^2$.
63. $12-x-x^2$.
64. $35+2y-y^2$.
65. $1-x-42x^2$.
66. $3x^2+9x+6$.
67. $10x^2+110x+280$.
68. $x^2y^2+8xy-9$.
69. $28-3ab-a^2b^2$.
70. $2x^2-6x-36$.
71. $5x^2-10x+5$.
72. $x^2-5xy+6y^2$.
73. $x^2-7xy-18y^2$.
74. $x^2+10xy-75y^2$.

Example V. Simplify $\dfrac{x^2+x-2}{x^2-x-6}$.

$$\frac{x^2+x-2}{x^2-x-6}=\frac{(x-1)(x+2)}{(x+2)(x-3)}=\frac{\mathbf{x-1}}{\mathbf{x-3}}.$$

Example VI. Simplify $\dfrac{1}{x-2}-\dfrac{3}{x^2-x-2}$.

$$\frac{1}{x-2}-\frac{3}{x^2-x-2}=\frac{1}{x-2}-\frac{3}{(x+1)(x-2)}$$

$$=\frac{(x+1)-3}{(x+1)(x-2)}=\frac{x-2}{(x+1)(x-2)}$$

$$=\frac{\mathbf{1}}{\mathbf{x+1}}.$$

Example VII. Solve $\frac{x}{x-4}-\frac{x-1}{x+1}=0$.

Multiply each side by $(x-4)(x+1)$.

$$\therefore\ x(x+1)-(x-1)(x-4)=0.$$
$$\therefore\ x^2+x-(x^2-5x+4)=0;$$
$$\therefore\ x^2+x-x^2+5x-4=0; \quad \therefore\ 6x-4=0;$$
$$\therefore\ 6x=4; \quad \therefore\ x=\tfrac{4}{6}=\tfrac{2}{3}.$$

Exercise XII. *b*.

1. Simplify $\dfrac{x^2+8x+15}{x^2+5x}$.
2. What is the L.C.M. of x^2-2x+1 and x^2-x?
3. What is the H.C.F. of x^2+2x and $x^2+7x+10$?
4. What is the quotient if x^2-6x+8 is divided by $x-2$?
5. Fill in the blanks in $(x+2)(x+\quad)\equiv(x^2\ldots+14)$.
6. Fill in the blanks in $(x-5)(x+\quad)\equiv(x^2\ldots-15)$.
7. If $x+3$ is a factor of $x^2+ax+15$, what is a?
8. If $x-4$ is a factor of x^2+bx-4, what is b?
9. If $x+7$ is a factor of x^2+9x+c, what is c?
10. If $x+7$ is a factor of x^2+6x+d, what is d?
11. What is the L.C.M. of $2a+2$ and a^2-4a-5?
12. Simplify $\dfrac{t-4}{t^2-3t-4}$.
13. Simplify $\dfrac{1}{y-2}-\dfrac{1}{y+1}$.
14. Find y if $\dfrac{1}{y-2}-\dfrac{1}{y+1}=\dfrac{3}{y^2}$.
15. Find t if $\dfrac{1}{t+1}+\dfrac{1}{t-2}=\dfrac{2}{t}$.
16. Simplify (i) $1-\dfrac{1}{t+1}$; (ii) $\dfrac{1}{a+b}+\dfrac{1}{a-b}$.
17. What is the coefficient of x in

    (i) x^2-7x-5; (ii) $ax^2+bx+x+c$; (iii) $(2x-1)(3x+2)$?

Simplify the expressions and solve the equations in Nos. 18-28 :

18. $\dfrac{x^2+2x+1}{x^2+5x+4}$.

19. $\dfrac{2p}{p^2+7p+12}-\dfrac{1}{2p+6}$.

20. $\dfrac{z}{z+2}=\dfrac{2z-1}{2z+1}$.

21. $\dfrac{b+1}{b+2}-\dfrac{b+2}{b+3}$.

22. $\dfrac{2}{y^2-2y-15}-\dfrac{1}{y^2-5y}$.

23. $\dfrac{3y+4}{3y}=1+\dfrac{3}{y+7}$.

24. $1-\dfrac{x+1}{(x+2)(x+3)}$.

25. $\dfrac{x^2-4x-5}{x^2-2x-15}\div\dfrac{x^2-x-2}{x^2-x-12}$.

26. $\dfrac{6(t+2)}{3t-1}=\dfrac{2t}{t-5}$.

27. $\dfrac{3y}{x-3y}-\dfrac{y}{x-y}$.

28. $\dfrac{1}{(x+1)(x+2)}-\dfrac{2}{(x+1)(x+2)(x+3)}$.

SUPPLEMENTARY EXERCISE. S. 7

1. One of the factors of $x^3+6x^2+11x+6$ is $x+1$; find the others. What is the value of $x^3+6x^2+11x+6$ when $x=-1$?

2. Fill in the blanks in $\dfrac{2}{x-3}\equiv\dfrac{\quad}{x^2-4x+3}$.

3. What is the quotient if $x^2-45x+200$ is divided by $x-5$?

4. What is the coefficient of x in (i) $a+x-ax-a^2$;

   (ii) $(x+a)^2-(x+1)^2$; (iii) $(ax+b)\left(cx+\dfrac{1}{a}\right)$?

5. Fill in the blank in $\dfrac{x}{x+1}\equiv\dfrac{x^2-x}{\quad}$.

6. Simplify $2\div\left(1+\dfrac{x-1}{x+1}\right)$.

7. Simplify $\left(1-\dfrac{a}{a+4}\right)\left(\dfrac{a}{a-3}-1\right)$.

8. Solve $\dfrac{3}{z+1}+\dfrac{8}{z+5}=\dfrac{11}{z}$.

9. Solve $$\frac{4}{2x-1}-\frac{1}{x+1}=\frac{5-x}{(2x-1)(x+1)}.$$

10. Simplify $$\left(1+\frac{a+b}{a-b}\right)\div\left(1+\frac{a-b}{a+b}\right).$$

11. Simplify $$\frac{x+1}{x^2-x-12}-\frac{x+3}{x^2+x-6}.$$

12. Solve $$\frac{x+7}{x^2-x-6}-\frac{x-5}{x^2-4x+3}=0.$$

13. Simplify $$\frac{2}{x^2-4x}-\frac{1}{x^2-6x+8}.$$

14. Solve $$\frac{3x+2}{x^2+5x+6}-\frac{2}{x+2}=\frac{3}{x+3}.$$

15. Simplify $$\frac{1}{x(x+1)}+\frac{1}{(x+1)(x+2)}+\frac{1}{x(x+2)}.$$

Factors of ax^2+bx+c.

First consider the reverse process :

$$\begin{aligned}(ex+f)(px+q)&=ex(px+q)+f(px+q)\\&=epx^2+eqx+fpx+fq\\&=epx^2+(eq+fp)x+fq.\end{aligned}$$

In order to work backwards, we must take the term in x and break it up into two terms whose product equals the product of the other two terms, since

$$eqx\times fpx=efpqx^2=epx^2\times fq.$$

This process is best understood by taking some numerical examples.

Example VIII. Factorise $8x^2+10x+3$.

Replace $10x$ by two terms whose product is

$$8x^2\times 3=24x^2=4x\times 6x.$$

$$\begin{aligned}\therefore\ 8x^2+10x+3&=8x^2+4x+6x+3\\&=4x(2x+1)+3(2x+1)\\&=(2x+1)(4x+3).\end{aligned}$$

Note. Check by multiplying by inspection.

Example IX. Factorise $12x^2 - x - 35$.

Replace $-x$ by two terms whose product is

$$(12x^2)(-35) = (-3 \times 4 \times 5 \times 7x^2) = (+20x)(-21x).$$

$$\therefore\ 12x^2 - x - 35 = 12x^2 + 20x - 21x - 35$$
$$= 4x(3x+5) - 7(3x+5)$$
$$= (\mathbf{3x+5})(\mathbf{4x-7}).$$

Example X. Factorise $6x^2 + 11xy - 10y^2$.

Replace $11xy$ by two terms whose product is

$$(6x^2)(-10y^2) = -60x^2y^2 = (-4xy)(+15xy).$$

$$\therefore\ 6x^2 + 11xy - 10y^2 = 6x^2 - 4xy + 15xy - 10y^2$$
$$= 2x(3x-2y) + 5y(3x-2y)$$
$$= (\mathbf{3x-2y})(\mathbf{2x+5y}).$$

EXERCISE XII. *c*.

Write down the following products, Nos. 1-9:

1. $(2x+3)(x+5)$. 2. $(2x-3)(x+5)$. 3. $(2x+3)(x-5)$.
4. $(2x-3)(x-5)$. 5. $(3-5x)(4+3x)$. 6. $(3+5x)(4-3x)$.
7. $(3-5x)(4-3x)$. 8. $(2a-7b)(a+4b)$. 9. $(3a+5b)(3a-5b)$.

Write down the coefficient of x in Nos. 10-12:

10. $(7x-3)(4x+11)$. 11. $(2x-5)(5x-9)$. 12. $(4x-7)(6x+3)$.

Write down the constant term in Nos. 13-15:

13. $(3-9x)(5x-4)$. 14. $(7-3x)(3x+7)$. 15. $(2x-5)^2$.

Factorise Nos. 16-41, check your answers, mentally.

16. $3x^2+7x+2$. 17. $3x^2-7x+2$. 18. $2x^2+5x+2$.
19. $2x^2-5x+2$. 20. $5x^2+8x+3$. 21. $2x^2+11x+14$.
22. $3x^2+x-2$. 23. $3x^2-x-2$. 24. $5x^2-2x-7$.
25. $6x^2+7x-5$. 26. $15x^2-11x+2$. 27. $6x^2+5x-6$.
28. $3x^2+14x-5$. 29. $8a^2-14a+3$. 30. $6y^2-4y-2$.
31. $2x^2+11xy-21y^2$. 32. $3-5x+2x^2$. 33. $2-x-15x^2$.
34. $14a^2+29a-15$. 35. $2+7y-15y^2$. 36. $2x^2-xy-21y^2$.

37. $15+14b-8b^2$. 38. $20a^2+44a-15$. 39. $12x^2-2xy-30y^2$.

40. $4x^4+13x^2y^2+9y^4$. 41. $6x^4+10x^2yz-4y^2z^2$.

42. Find the L.C.M. of $2x^2+3xy-2y^2$ and $10x^2-7xy+y^2$.

43. Write down the product of $2a+21b$ and $3a-10b$.

44. Simplify $\dfrac{x}{2x-3}-\dfrac{x+2}{2x-1}$.

45. Find the H.C.F. of $5x+10$ and $5(2x^2-5x-18)$.

46. Factorise $(2x-3)(3x-5)-(2x-3)(x-2)$.

47. Simplify $\dfrac{1}{x-3}-\dfrac{2x+5}{3x^2-7x-6}$.

48. If $\dfrac{2x-3}{3x-11}=\dfrac{y}{3x^2-8x-11}$, what is y?

49. If $6x^2-29x-65$ is divided by $3x+5$, what is the quotient?

50. Factorise $(2x^2-5x+2)(7+5x-2x^2)$.

51. $x-1$ is one factor of $2x^3+7x^2+x-10$; find the others. What is the value of $2x^3+7x^2+x-10$ when $x-1=0$?

52. Solve $\dfrac{x}{2x-1}+\dfrac{3}{2x+2}=\dfrac{1}{2}$.

53. Simplify $\dfrac{2}{3x^2+x-2}-\dfrac{1}{4x^2+3x-1}$.

54. Simplify $(3x^2-x-10)(2x^2+5x-3)\div(6x^2+7x-5)$.

55. Find a square root of $(a^2-5a+6)(a^2+3a-10)(a^2+2a-15)$.

Factorise the following:

56. $6x^2-29x+35$. 57. $12x^2+7xy-10y^2$.

58. $6+a-12a^2$. 59. $15b^2-17b-42$.

60. $18-9c-2c^2$. 61. $14y^2+33yz-5z^2$.

62. $8x^2+22x+5$. 63. $6a^2-31ab+35b^2$.

64. $6-47x-8x^2$. 65. $15p^2-29p-14$.

66. $11x-2-15x^2$. 67. $6-27t+30t^2$.

68. $18a-9-8a^2$. 69. $13b-1-42b^2$.

70. $6x^2-25xy+24y^2$.

EXTRA PRACTICE EXERCISES. E.P. 10.

FACTORS.

Find the factors of the following expressions :

1. $ab - bc - ad + cd$.
2. $t^2 - 7t + 12$.
3. $p^2 - 3p - 4$.
4. $ax - 2bx + ay - 2by$.
5. $x^2 + 4x - 21$.
6. $y^2 + 10y + 24$.
7. $ab - 2ac + 2bd - 4cd$.
8. $3a^2 + 5a - 2$.
9. $1 - 11x + 24x^2$.
10. $b^2 - bc + bd - cd$.
11. $z^2 - 5z - 14$.
12. $2b^2 + b - 6$.
13. $xy - xz - y^2 + yz$.
14. $1 - 2t - 15t^2$.
15. $12a^2 - a - 6$.
16. $1 + a + a^2 + a^3$.
17. $10c^2 - 3c - 1$.
18. $10l^2 - 19lm + 6m^2$.
19. $x^2 - xy + yz - xz$.
20. $3t^2 + 9t - 30$.
21. $6 - p - 12p^2$.
22. $1 + xy - x - y$.
23. $a^2 - 16a + 48$.
24. $20b^2 - 49b + 30$.
25. $a(a-1) + a - 1$.
26. $4t^2 + 12t - 72$.
27. $15 + 14p - 8p^2$.
28. $1 - p - q(p-1)$.
29. $12y^2 - 7y + 1$.
30. $12x^2 - 16x - 3$.
31. $4ax - 6bx + 9by - 6ay$.
32. $3x^2 + xy - 2y^2$.
33. $6x^2 - 2xy + ay - 3ax$.
34. $12r^2 + 32rs + 5s^2$.
35. $x^2 + (a+b)x + ab$.
36. $28a^2 - 3ab - 18b^2$.
37. $3x^3 - 5x^2y + 6xy^2 - 10y^3$.
38. $21c^2 + 34cd + 8d^2$.
39. $x^2 - (p-q)x - pq$.
40. $20y^2 + 21yz - 54z^2$.
41. $1 + x^2 + xy + x^3y$.
42. $2a^4 + 3a^2bc - 2b^2c^2$.
43. $ax + 4 - 2x - 2a$.
44. $2x^4 + 11x^2 + 15$.
45. $(2a+b)(c+d) + c + d$.
46. $3 - 5t^2 - 12t^4$.
47. $x - y + (y-x)^2$.
48. $36a^2 + 27ab - 28b^2$.
49. $(c-d)^2 + c^2 - cd$.
50. $105 - 17p - 12p^2$.
51. $(x+1)^2 - (x+1)(2x-3)$.
52. $x(x+4y) - 3y(3x-2y)$.
53. $12x^2 + 5xy - 72y^2$.
54. $(x-y)^2 - 2x(x-y)$.
55. $a(x+y) - xy - a^2$.
56. $x(x+1) + (x+4)(x-3)$.
57. $a(b^2+c^2) - bc(1+a^2)$.
58. $x^2y^2 - 7xyz - 18z^2$.
59. $14a^2 - 13ab - 12b^2$.
60. $6x^2 - 4xy + 10yz - 15xz$.

The Difference of Two Squares.

By multiplication, we find that

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})=\mathbf{A}^2-\mathbf{B}^2 \text{ (see also p. 240).}$$

$\therefore$ the factors of A^2-B^2 are $A+B$ and $A-B$.

In this, A and B can stand for any numbers or any expressions.

Example XI. Factorise (i) x^2-9; (ii) $16x^2-49$; (iii) $(x+a)^2-b^2$.

(i) $x^2-9=x^2-3^2=(\mathbf{x}+\mathbf{3})(\mathbf{x}-\mathbf{3})$.

(ii) $16x^2-49=(4x)^2-7^2=(\mathbf{4x}+\mathbf{7})(\mathbf{4x}-\mathbf{7})$.

(iii) $(x+a)^2-b^2=\{(x+a)+b\}\{(x+a)-b\}$
$=\{\mathbf{x}+\mathbf{a}+\mathbf{b}\}\{\mathbf{x}+\mathbf{a}-\mathbf{b}\}$.

Exercise XII *d*.

1. What is the product of (i) $x+3$ and $x-3$; (ii) $x+y$ and $x-y$; (iii) $x+5y$ and $x-5y$; (iv) $3x+4y$ and $3x-4y$?
2. What is the quotient if (i) b^2-c^2 is divided by $b-c$; (ii) p^2-q^2 is divided by $p+q$; (iii) a^2-9b^2 is divided by $a-3b$; (iv) $4x^2-25y^2$ is divided by $2x+5y$?
3. Use factors to evaluate (i) 24^2-23^2; (ii) 22^2-20^2; (iii) 73^2-63^2; (iv) $2{\cdot}8^2-1{\cdot}2^2$.
4. (i) Show that $21\times19=20^2-1^2$ and evaluate it.
(ii) In the same way, evaluate 43×37 and 56×64.
5. Evaluate R^2-r^2 when $R=3\frac{1}{4}$ and $r=2\frac{1}{4}$.
6. Evaluate D^2-d^2 when $D=5{\cdot}8$ and $d=4{\cdot}2$.

Find the factors of the following, Nos. 7-30:

7. a^2-4.	8. b^2-49.	9. c^2-1.
10. $100-x^2$.	11. $4y^2-1$.	12. $4z^2-9$.
13. $9a^2-b^2$.	14. $4b^2-9c^2$.	15. $16t^2-25$.
16. a^2b^2-16.	17. a^2-25b^2.	18. $p^2q^2-r^2$.
19. a^4-16.	20. $3x^2-12$.	21. $5-20b^2$.
22. $(x+3)^2-5^2$.	23. $(x-4)^2-49$.	24. a^4-1.
25. $(a+b)^2-c^2$.	26. $(x+y)^2-1$.	27. $(2a+3b)^2-b^2$.
28. $a^2-(b+c)^2$.	29. $x^2-(y-z)^2$.	30. $4-9(a+b)^2$.

31. Evaluate $\pi(R^2-r^2)$ when $\pi=3{\cdot}14$, $R=5{\cdot}25$, $r=4{\cdot}75$; interpret geometrically.

32. Write down the product of $3a^4+7$ and $3a^4-7$.

33. Find the H.C.F. of x^2-x and x^2-1.

34. Simplify $$\frac{(p+3)^2}{p^2-9}.$$

35. Find the L.C.M. of $xy-y$ and x^2-1.

36. Find the L.C.M. of $4a^2$ and $4a^2-4$.

37. What is the coefficient of x in $(3x+7)(3x-7)$?

38. Is x^2+5 a factor of x^4+25?

39. Simplify $$\frac{a}{x-a}-\frac{a}{x+a}.$$

40. Solve $(x+5)(x-5)+(x+6)(x-6)=x(2x+1)$.

41. Simplify $$\frac{a^2-\dfrac{1}{a^2}}{a-\dfrac{1}{a}}\times a^2.$$

42. Write down the product of $4x^2+1$ and $4x^2-1$.

43. What is the L.C.M. of x^2+2x-3 and x^2-9?

44. Simplify $$\left(x-\frac{xy}{x+y}\right)\left(x+\frac{xy}{x-y}\right).$$

45. Divide 83^2-27^2 by 11.

Squares and Square Roots.

$$(A+B)^2=(A+B)(A+B)=A^2+AB+AB+B^2;$$

$$\therefore\ (\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+\mathbf{2AB}+\mathbf{B}^2 \text{ (see also p. 243).}$$

This gives a rule for writing down the square of any expression which consists of two terms.

Thus $$(3x+5y)^2=(3x)^2+2(3x)(5y)+(5y)^2$$
$$=9x^2+30xy+25y^2$$
and $$(3x-5y)^2=(3x)^2+2(3x)(-5y)+(-5y)^2$$
$$=9x^2-30xy+25y^2.$$

It is important to be able to recognise a perfect square when you have to deal with it.

$$25x^2-40xy+16y^2=(5x)^2-2(5x)(4y)+(4y)^2$$
$$=(5x-4y)^2=(4y-5x)^2.$$

Notice that an expression has always *two* square roots.

Thus $49=(+7)^2=(-7)^2$; $\therefore$ the square root of 49 is either $+7$ or -7.

And the square root of $25x^2-40xy+16y^2$ is either $5x-4y$ or $4y-5x$, *i.e.* $+(5x-4y)$ or $-(5x-4y)$.

It may help the reader to see the following relations together:

$$(A+B)^2=A^2+2AB+B^2,$$
$$(A-B)^2=A^2-2AB+B^2,$$
$$(A+B)(A-B)=A^2-B^2.$$

Example XII. What number must be added to x^2+6x to make the result a perfect square? Of what expression is the sum the square?

Draw a rectangle $ABCD$, having $AB=(x+6)$ in., $AD=x$ in.; then its area is (x^2+6x) sq. in. In Fig. 229, $AP=x$ in., $PB=6$ in.

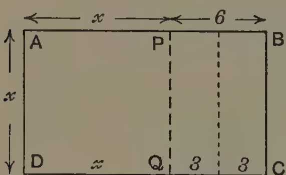

FIG. 229.

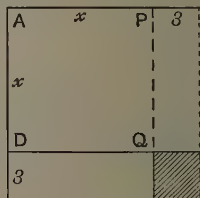

FIG. 230.

Transpose half of the rectangle $PBCQ$ and fit it on to DQ, as in Fig. 230; we then have a square of side x in., bordered with two rectangles, each of width $\frac{1}{2}$ of $6=3$ in. In order to complete the square we must add the shaded area in Fig. 230, a square of side 3 in. and of area 3^2 sq. in. The new area then obtained is a square of side $(x+3)$ in. and area $(x+3)^2$ sq. in. $=(x^2+6x+9)$ sq. in.

$\therefore$ to x^2+6x, we must add $(\frac{1}{2}$ of $6)^2=3^2=9$; and the sum is then equal to $(x+3)^2$.

The sum is therefore the square of $x+3$ or of $-(x+3)$.

Example XIII. What number must be added to $49x^2-126x$, to make the result a perfect square? Of what expression is the sum a square?

$$(7x-c)^2=49x^2-14cx+c^2.$$

We must choose c, so that $14c=126$; $\therefore$ $c=\frac{126}{14}=9$.

$$\therefore\ (7x-9)^2=49x^2-126x+81.$$

$\therefore$ we must add 81; the sum is then the square of $7x-9$ or of $-(7x-9)$.

Exercise XII. *e*.

Write down the squares of the expressions in Nos. 1-12 :

1. $x+3$. 2. $x-1$. 3. $a+4b$. 4. $3a-b$.
5. $2x-3y$. 6. $5x^2+7y$. 7. $2xy+z$. 8. x^2+y^2.
9. $B-A$. 10. A^2-B^2. 11. B^2-A^2. 12. $a+\frac{1}{a}$.

Write down two square roots of each of the following, Nos. 13-18 :

13. x^2-2x+1. 14. $a^2+6ab+9b^2$. 15. $4-4y+y^2$.
16. $4p^2+20pq+25q^2$. 17. $100+60t+9t^2$. 18. $x^2+x+\frac{1}{4}$.

What must be added to the following to make the result a perfect square ? Of what expression is the sum the square ? Illustrate Nos. 19, 20 geometrically.

19. x^2+8x. 20. a^2+5a. 21. y^2-10y.
22. b^2-7b. 23. $4x^2+28x$. 24. $9y^2-42y$.
25. x^2+2px. 26. x^2+ax. 27. y^2-by.

Fill in the missing terms in the following, Nos. 28-35 :

28. $(x+5)^2=x^2\ldots+25$. 29. $(2x+7)^2=4x^2\ldots+49$.
30. $(x-3a)^2=x^2\ldots+9a^2$. 31. $(2x-a)^2=4x^2\ldots+a^2$.
32. $(x\ldots)^2=x^2-12x+36$. 33. $(2x\ldots)^2=4x^2-28x+49$.
35. $(a\ldots)^2=a^2+8ab\ldots$. 35. $(\ldots-3a)^2=4\ldots+9a^2$.

In the following, Nos. 36-43, state whether the expression is a perfect square ; if it is, give two square roots of it.

36. $x^2+4xy+4y^2$. 37. $x^2+6xy+36y^2$.
38. $4a^2-40ab+25b^2$. 39. $p^2-p+\frac{1}{4}$.
40. $x^4+6x^2y^2+9y^4$. 41. $a^2-2ab-b^2$.
42. $1-2y+y^2$. 43. $x^2+\frac{1}{x^2}+2$.

44. What is the coefficient of x in $(3x-7)^2$?
45. What is the constant term in $5(2x+3)^2$?
46. Give two terms, either of which, when added to x^2+49, makes the result a perfect square.
47. Simplify $\dfrac{(3x-3)^2}{(x-1)^2}$.

Miscellaneous Factors.

Exercise XII. *f*.

Factorise the following expressions :

1. $3a^2 - 3a$.
2. $p^2 + 9p + 8$.
3. $x^2y - xy^2$.
4. $t - 4t^2$.
5. $x^4 - y^2$.
6. $l^2 - 15l + 26$.
7. $5n^2 - 25$.
8. $y^2 + y - 90$.
9. $a(b+c) - e(c+b)$.
10. $1 + t + t^2 + t^3$.
11. $y^2 - 64z^2$.
12. $x^2 + 13xy + 40y^2$.
13. $a^3b - ab^3$.
14. $3t^2 + 5t - 2$.
15. $x(a-b) + y(b-a)$.
16. $x^4 - 4x^2z^2$.
17. $7x^2 - 28z^2$.
18. $y^3 + 3y^2 + 2y$.
19. $4x^2 - xy - 5y^2$.
20. $a^2 - ab + bd - ad$.
21. $x^4 - 4x^2 + 3$.
22. $t^3 - t^2 + t$.
23. $x^2 - x(y+z) + yz$.
24. $a^2b^2 - 2ab - 35$.
25. $p^2 + 4pq + 4q^2$.
26. $y^4 - 36y^2$.
27. $a^3b + a^2b^2 + ab^3$.
28. $10a^2 + 50ax + 60x^2$.
29. $ab + xy - ay - bx$.
30. $1 - 5t + 4t^2$.
31. $ab - a - b + 1$.
32. $4x^2 + 20xz + 25z^2$.
33. $3p^2 + 9p - 120$.
34. $x^2 - pq - (p-q)x$.
35. $72 - 17h + h^2$.
36. $cx - dx + dq - cq$.
37. $17y - 72 - y^2$.
38. $(a-b)^2 - 100$.
39. $a^3 - a^2b + ab^2 - b^3$.
40. $mp^2 - mq^2 + lq^2 - lp^2$.
41. $5x^2 - 10xy + 5y^2$.
42. $x^2 - 1 + (x+1)^2$.
43. $m(n-1) + (n-1)^2$.
44. $x^2 - (a+3)x + 3a$.
45. $z^2 - 16(x-y)^2$.

EXTRA PRACTICE EXERCISES. E.P. 11.

Fractions.

Simplify the following expressions :

1. $\dfrac{3x-3y}{3x+3y}$.
2. $\dfrac{2a+2b}{5a+5b}$.
3. $\dfrac{a^2-ax}{ax-x^2}$.
4. $\dfrac{c+c^2}{3+3c}$.
5. $\dfrac{p^2-2p}{p^2+2p}$.
6. $\dfrac{x^2-xy}{x^2-y^2}$.
7. $\dfrac{ab^2+a^2b}{abc}$.
8. $\dfrac{2x-2y}{(x-y)^2}$.
9. $\dfrac{4x-4y}{6y-6x}$.
10. $\dfrac{a^2-b^2}{(a-b)^2}$.
11. $\dfrac{6x^2-6xy}{4xy}$.
12. $\dfrac{3x-3}{(1-x)^2}$.
13. $\dfrac{b^3-b^2c}{b^2c-bc^2}$.
14. $\dfrac{(a-2)^2}{(2a-4)^2}$.
15. $\dfrac{2x-2y}{3x^2y-3xy^2}$.

16. $\dfrac{x-3}{x^2-x-6}$.

17. $\dfrac{a^2+a-6}{2a+6}$.

18. $\dfrac{ab-b^2}{a^2+ab-2b^2}$.

19. $\dfrac{x^2+6x+8}{x^2-4}$.

20. $\dfrac{x^2+xy-2y^2}{x^2-2xy+y^2}$.

21. $\dfrac{x^2-8x+15}{x^2+2x-35}$.

22. $\dfrac{y^2-2y+1}{y^2+3y-4}$.

23. $\dfrac{x^2-x}{(2x-2)^2}$.

24. $\dfrac{x^2+x-2}{x^2-1}$.

25. $\dfrac{x^2-1}{x}\times\dfrac{x^2}{x^2+2x+1}$.

26. $\dfrac{a^2+a-2}{a^2-a-6}\times\dfrac{a-3}{a-2}$.

27. $\dfrac{x^2+xy}{xy-xz}\times\dfrac{y^2-yz}{xz+yz}$.

28. $\dfrac{4x-12}{3x}\times\dfrac{9x^2}{6x-18}$.

29. $\dfrac{2x}{3x-3}\div\dfrac{4y}{x^2-x}$.

30. $\dfrac{5a}{a-3b}\div\dfrac{2a}{3a-b}$.

31. $\dfrac{x^2-x}{ax-a}\div\dfrac{ax}{ax-x}$.

32. $\dfrac{x^2-xy}{xy+y^2}\div\dfrac{xy-y^2}{x^2+xy}$.

33. $\dfrac{x^2-2xy+y^2}{x^2-y^2}\times\dfrac{x^2+2xy+y^2}{x^2+xy}$.

34. $\dfrac{b^2-c^2}{b^2-2bc+c^2}\div\dfrac{c}{b^2-bc}$.

35. $\dfrac{b^2+2bc+c^2}{b^2+2bc}\times\dfrac{b}{b^2-c^2}$.

36. $\dfrac{a^2-4ab+4b^2}{a^2+ab-6b^2}\div\dfrac{a^2+3ab}{a^2+6ab+9b^2}$.

37. $\dfrac{x}{x-2}-\dfrac{2}{x+2}$.

38. $\dfrac{2x-3}{3x-9}-\dfrac{x-2}{2x-6}$.

39. $\dfrac{1}{3a-3b}+\dfrac{1}{6a-6b}$.

40. $\dfrac{a}{a-b}-\dfrac{b}{a+b}$.

41. $\dfrac{4}{x+3}+\dfrac{12}{x^2-9}$.

42. $\dfrac{x}{x-y}+\dfrac{y}{y-x}$.

43. $\dfrac{x+2}{x+1}-\dfrac{x+1}{x+2}$.

44. $\dfrac{ab}{a^2-b^2}-\dfrac{b}{a+b}$.

45. $\dfrac{1}{x^2-4}-\dfrac{1}{x^2+2x-8}$.

46. $\dfrac{a}{a-b}+\dfrac{a}{b-a}$.

47. $\dfrac{3}{x-2}-\dfrac{6}{x^2-4}$.

48. $\dfrac{a+2b}{a^2+ab}+\dfrac{a-b}{ab+b^2}$.

49. $\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x+y}{x-y}$.

50. $\dfrac{x}{x^2-4x+3}-\dfrac{2}{x^2-x-6}$.

51. $\dfrac{2b}{b^2-a^2}+\dfrac{1}{a-b}$.

52. $\dfrac{2b}{b+c}+\dfrac{c}{b-c}-\dfrac{2b^2}{b^2-c^2}$.

SUPPLEMENTARY EXERCISE. S. 8

1. Factorise (i) $25x^6 - y^6$; (ii) $4(a+3b)^2 - 9$; (iii) $n^5 - n$.
2. Divide $122^2 - 81^2$ by 41.
3. What is p if $\dfrac{2}{x+2} = \dfrac{p}{x^2-4}$?
4. Simplify $\dfrac{a}{a+b} + \dfrac{a}{a-b} - \dfrac{a^2+b^2}{a^2-b^2}$.
5. Simplify (i) $(x-y)^2 - (y-x)^2$; (ii) $\dfrac{(7x+14)^2}{(x+2)^2}$.
6. Solve $\dfrac{x+1}{x-2} - \dfrac{x-1}{x+2} = \dfrac{6}{x+1}$.
7. Find the L.C.M. of $(2x-2)^2$ and $(3x-3)^2$.
8. Express 22×24 as the difference of two squares.
9. Show that $2x+1$ is one square root of $4x^2+4x+1$; what is the other squar roo† ?
10. Factorise (i) $ah + bh + ch + ak + bk + ck$;
    (ii) $x(2y-3z) + 3z - 2y$.
11. Simplify $(5x-15)^2 - 25(x-3)^2$.
12. Is $x^2 + 5x + 6\frac{1}{4}$ a perfect square?
13. Simplify $\dfrac{1}{2x+4} + \dfrac{2}{3x+6} - \dfrac{x-3}{x^2-4}$.
14. Solve $\dfrac{x}{2x+1} + \dfrac{x}{2x-1} = 1 + \dfrac{2x-3}{4x^2-1}$.
15. Factorise (i) $x^3 - x^2 - x + 1$; (ii) $a^2b^2 + c^2d^2 - a^2c^2 - b^2d^2$.
16. For what values of x is $(x-29)^2 = 9^2$?

MISCELLANEOUS EXAMPLES

M. IV

1. What is the quotient if $xy(x-1)$ is divided by
   (i) x; (ii) $x(x-1)$?
2. What are the factors of $x^2 - x - 42$?
3. What is the product of $x+8$ and $x-8$?
4. Write down the product of $x-1$ and $x-13$.
5. Find the H.C.F. and L.C.M. of $x^2 - x - 20$ and $x^2 - 25$.

6. Simplify $$\frac{1}{x}-\frac{1}{x+5}.$$

7. Express $\frac{x}{x-6}$ as a fraction with $x^2-9x+18$ as denominator.

8. Multiply x^2+7x+6 by $x-3$. What are the factors of the product ?

9. What is the meaning of $4(x-1)^2$? What is the quotient if it is divided by $x-1$?

10. Express $\frac{2}{x-1}$ and $\frac{3}{x-2}$ as fractions with the same denominators.

11. What are the factors of 23^2-36 ?

12. Simplify $$\frac{x^2-4x}{2(x^2-2x-8)}.$$

13. Divide a^3-ab^2 by (i) $a(a-b)$, (ii) $a-b$.

14. $x-3$ is one factor of $x^3-7x^2-9x+63$. What are the others ?

15. Solve the equation $(2x+1)(6x-7)=(3x+4)(4x-6)$.

16. What is the quotient if x^3-x is divided by x^2-x ?

17. Simplify $$\frac{1}{x^2-x}-\frac{1}{x^3-x}.$$

18. (i) Factorise $2x^2-15x+18$;
(ii) If $x=6$, show that $2x^2-15x+18$ is equal to 0.

19. Write down the product of $2x-7$ and $x-2$.

20. Solve the equation
$$\frac{12x}{x+2}=6+2\left(\frac{3x+2}{x+1}\right).$$

21. What is the quotient if 100^2-93^2 is divided by 7 ?

22. Simplify $$x-\frac{x^2-3}{x+3}.$$

23. What number must be added to x^2+14x to make the result a perfect square, and of what is it then the square ?

24. Simplify $$\frac{x}{2(x+3)}-\frac{x}{2(x-3)}.$$

25. Solve the equation $\frac{6}{3x-5}-\frac{1}{x-5}=\frac{2}{2x-5}$.

26. Simplify $$\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x-1}\right).$$

27. Prove that $\left(1-\frac{1}{x}\right)^2=\frac{1-2x+x^2}{x^2}$.

28. Prove that $(x^2+5x+6)(x^2-5x+6)=(x^2-4)(x^2-9)$.

29. Simplify $(x+2)\div\left(1+\frac{2}{x}\right)$.

30. Simplify $\left(1-\frac{1}{x}\right)\div\left(x-\frac{1}{x}\right)$.

31. Solve the equation $\frac{2x-3}{3x-4}=\frac{4x-5}{6x-7}$.

32. What is the square root of $(x^2-2x+1)(x^2+4x+4)$?

33. Prove that $(x^2+x-2)(x^2-4x+3)(x^2-x-6)$ is a perfect square, and find its square root.

34. Divide x^3-4x by x^2+2x.

35. Prove that $(x-1)(x+1)(x^2+1)(x^4+1)=x^8-1$.

36. Factorise ab^2c-9ac^3.

37. Solve the equation $\frac{x-4}{x-2}+\frac{x+3}{x-5}=2$.

38. What is the least expression by which $(x^2-4x+3)(x^2+x-2)$ must be multiplied in order to make it a perfect square?

39. Simplify $\frac{1}{1-\frac{1}{x}}+\frac{1}{\frac{1}{x}-1}$.

40. Multiply $3x+7y-800$ by $3x+7y+800$.

41. Factorise $(x-3)^3-a^2b^2(x-3)$.

42. Divide $(2x^2-7x+4)^2-(x^2-2x+5)^2$ by x^2-5x-1.

43. Factorise $ab-7a+5b-35$.

44. Solve the equation $\frac{4(x+5)}{x+6}+\frac{3(x+6)}{x+5}=7$.

45. If $x+4$ is a factor of $x^2+ax-28$, what is the value of a and what is the other factor?

46. Write down the square of $3b-2c$.

47. Solve the equation $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$.

48. Simplify $\frac{x^2-x-2}{x^2-5x+6}-\frac{x^2+4x-5}{x^2+2x-15}$.

49. Evaluate $\dfrac{2240 \times 48^2}{64} - \dfrac{2240 \times 18^2}{64}$.

50. What single term must be added to $x^2 - 11ax$ in order to make the result a perfect square ?

51. Factorise $a^2 + 4 + \dfrac{4}{a^2}$.

52. Divide $x + \dfrac{1}{x} - 2$ by $\dfrac{1}{2x} - \dfrac{1}{2}$.

53. Solve the equation $\dfrac{x-4}{x+4} + \dfrac{x-6}{x+6} = 2$.

54. Factorise $a(a-1) - b(b+1)$.

55. Simplify $\dfrac{a^2 + 2ab + b^2}{a+b} + \dfrac{a^2 - 2ab + b^2}{a-b}$.

56. Factorise $(p-q)^2 + (p+x)^2 - (x-y)^2 - (q+y)^2$.

CHAPTER XIII

QUADRATIC EQUATIONS

Graphs of Quadratic Functions.

We have already discussed in Chapter VI. the graphical representation of functions, and the reader will find on p. 154 the graph of the function $11 + 24n - 3n^2$. The graph, there given, is a smooth curve, called a parabola, and it will be found that the graphs of all quadratic functions, *i.e.* functions of the form $ax^2 + bx + c$, are curves of the same kind; the actual shape depends on the scales selected for the two axes and on the coefficient of x^2.

In the function, $11 + 24n - 3n^2$, the coefficient of the term of the second degree is negative, -3; consequently the graph has a *highest* point and the function has a *greatest* (or maximum) value. In Example II. on p. 274, the coefficient of x^2 is positive and we there see (Fig. 233) that the graph has then a *lowest* point and the function has a *least* (or minimum) value. But the two curves are of the same nature; if we turn Fig. 158, p. 154, upside down, it becomes similarly situated to Fig. 233.

Example I. A marble is projected up a sloping groove and moves so that it is passing now (at zero hour, $t=0$) a marked

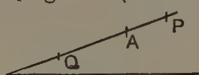

FIG. 231.

point A and t seconds later is s feet up the groove from A where $s = 20t - 5t^2$. Draw a graph showing the position of the marble from 2 seconds ago ($t = -2$) to 6 seconds ahead ($t = +6$).

Construct a table of values, as on p. 154.

t	-2	-1	0	1	2	3	4	5	6
$20t$	-40	-20	0	20	40	60	80	100	120
$5t^2$	20	5	0	5	20	45	80	125	180
$20t - 5t^2$	-60	-25	0	15	20	15	0	-25	-60

Plotting these values we obtain Fig. 232.

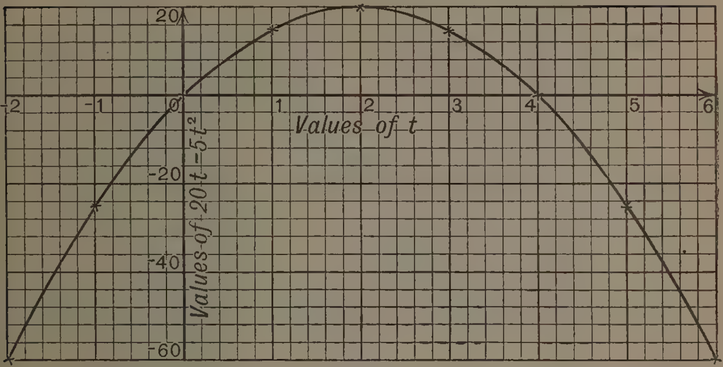

FIG. 232.

We can now use this graph to read off the position of the marble at any time represented in the figure and to read off the time at which the marble occupies any position within the range of the drawing.

EXERCISE XIII. *a*. (Oral.)

Use Fig. 232 to answer the following questions, Nos. 1-10.

1. How far above or below A is the marble when $t = 0{\cdot}6$, $3{\cdot}8$, $-0{\cdot}6$, $-1{\cdot}4$, $3{\cdot}5$, $2{\cdot}6$, $1{\cdot}4$, $-1{\cdot}8$, $-0{\cdot}3$ and explain what these values of t mean.

2. What is the highest point the marble reaches, and at what time ?

3. When is the marble at P (see Fig. 231), if AP is (i) 10 feet, (ii) 14 feet, (iii) 20 feet, (iv) 4 feet ?

4. When is the marble at Q (see Fig. 231), if AQ is (i) 16 feet, (ii) 34 feet, (iii) 52 feet, (iv) 60 feet ?

5. At what time does the marble pass A, coming down ?

6. For what values of t is $20t - 5t^2$ equal to 14 ? What does this mean ? Solve $20t - 5t^2 = 14$.

7. Does the marble reach a point 25 feet above A? Is there a value of t for which $20t - 5t^2 = 25$?

8. Find, when possible, approximate values of t such that

(i) $20t - 5t^2 = 4$; (ii) $20t - 5t^2 = -14$;

(iii) $5t^2 - 20t = 40$; (iv) $5t^2 - 20t + 8 = 0$.

9. If, in Fig. 231, AQ is 60 feet, how long is it between the times when the marble passes Q going up and coming down?

10. Can you interpret the fact that the graph is (i) steeper at $t = -2$ than at $t = 0$; (ii) flat at $t = 2$?

11. A marble rolling in a sloping groove passes at zero hour a marked point B and t seconds later is s feet *down* the groove from B where $s = t^2 - 3t$. Make a table of values of s for values of t from -2 to $+5$ and plot the graph showing the positions of the marble for this interval of time. Why should you include $t = 1{\cdot}5$ in your table? Use your graph to answer the following questions:

(i) How far above or below B is the marble when $t = 0{\cdot}5$, $2{\cdot}5$, $-1{\cdot}5$, $4{\cdot}5$, $-1{\cdot}7$, $-0{\cdot}8$.

(ii) What is the highest point the marble reaches and at what time?

(iii) When is the marble (*a*) 4 ft. below B, (*b*) 1 ft. above B, (*c*) 8 ft. below B, (*d*) $1{\cdot}8$ ft. above B?

(iv) At what time does the marble pass B coming down?

(v) Find approximate values of t such that (*a*) $t^2 - 3t = 6$, (*b*) $t^2 - 3t = -1$, (*c*) $t^2 - 3t = 3$, (*d*) $t^2 - 3t + 1{\cdot}5 = 0$.

Example II. Draw the graph of $2x^2 - 7x - 2$ for values of x from $x = -1$ to $x = +4$.

Construct a table of values, as on p. 272.

x	-1	0	1	$1{\cdot}5$	2	3	4
$2x^2$ $7x$	2 -7	0 0	2 7	$4{\cdot}5$ $10{\cdot}5$	8 14	18 21	32 28
$2x^2 - 7x - 2$	7	-2	-7	-8	-8	-5	2

Plotting these values, we obtain Fig. 233.

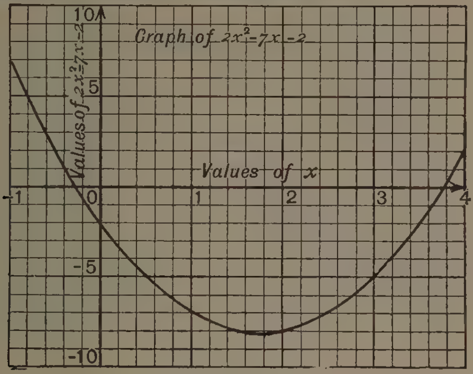

FIG. 233.

We can now use this graph to read off the value of the function $2x^2-7x-2$ for any value of x represented in the figure and to read off the value of x for which the function $2x^2-7x-2$ takes any assigned value, which comes within the range of the drawing.

EXERCISE XIII. *b.* (Oral.)

Use Fig. 233 to answer the following questions :

1. What is approximately the value of $2x^2-7x-2$ when $x=0{\cdot}4$, $1{\cdot}2$, $2{\cdot}5$, $2{\cdot}8$, $3{\cdot}5$, $3{\cdot}9$, $-0{\cdot}5$, $-0{\cdot}9$?

2. What is the least value of $2x^2-7x-2$? For what value of x is $2x^2-7x-2$ least ?

3. State approximately the values of x for which $2x^2-7x-2$ is equal to (i) 2, (ii) 1, (iii) -2, (iv) 0, (v) -5, (vi) -6.

4. For what value of x is $2x^2-7x-2$ equal to 5 ? Is there more than one answer ? If so, why cannot you find it ?

5. Is there a value of x for which $2x^2-7x-2$ equals -10 ?

6. Can you draw a line up the paper about which the curve is symmetrical ?

7. Solve the equation $2x^2-7x-2=0$.

8. Solve the equations :

(i) $2x^2-7x-2=1$; (ii) $2x^2-7x-2=-2$;

(iii) $2x^2-7x-2=-6$; (iv) $2x^2-7x-2=2$.

9. If $2x^2-7x+5=0$, what is the value of $2x^2-7x-2$? Hence solve $2x^2-7x+5=0$.

10. Solve, where possible, the equations :

(i) $2x^2-7x=-1$; (ii) $2x^2-7x=-4$;
(iii) $2x^2-7x=5$; (iv) $2x^2-7x=7$;
(v) $2x^2-7x=-6$; (vi) $2x^2-7x=-9$;
(vii) $7x-2x^2=3$; (viii) $7x-2x^2+6=0$.

EXERCISE XIII. c.

1. Draw the graph of $4x-x^2$ from $x=-1$ to $x=5$. Use it to find approximate values of x such that

(i) $4x-x^2=2$; (ii) $4x-x^2=-3$; (iii) $4x-x^2=2{\cdot}5$;
(iv) $4x-x^2=-1$; (v) $x^2-4x-4=0$; (vi) $x^2-4x+3=0$.

For what values of d has $4x-x^2=d$ two roots?

2. Draw the graph of x^2-5x+6 from $x=0$ to $x=5$. Find the roots of $x^2-5x+6=0$. Use the graph to solve

(i) $x^2-5x+1=0$; (ii) $x^2-5x+3=0$; (iii) $x^2-5x+4=0$.

Has x^2-5x+6 a greatest value or a least value, and how much is it?

3. Draw, on the same scale and axes used for No. 2, the graph of x^2-5x+7 from $x=0$ to $x=5$. Use the graph to solve

(i) $x^2-5x+7=4$; (ii) $x^2-5x+3=0$; (iii) $x^2-5x+2=0$.

Is there a value of x which makes x^2-5x+7 equal to 0? What is the least value of x^2-5x+7?

4. Fig. 234 shows a closed vessel, lowest point A, top point B, so shaped that when the depth of water in it is x feet,

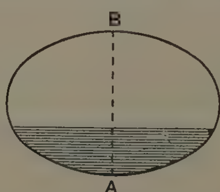

FIG. 234.

the area of the cross-section of the water-surface is $(8x-x^2)$ sq. ft.

Draw the graph of $8x - x^2$ for values of x from 0 to 8 and use the graph to answer the following questions :

(i) What is the cross-section of the surface when the depth is 2·5 ft., 3·6 ft., 4·4 ft., 6·8 ft. ?

(ii) What is the depth if the cross-section of the surface is 12 sq. ft., 14 sq. ft., 4 sq. ft., 8 sq. ft. ?

(iii) What is the length of AB ?

5. A farmer uses 200 yards of hurdle-fencing to enclose part of a field, one side of the rectangular enclosure being formed by part of a straight hedge, shown by the dotted line in Fig. 235. With the notation of Fig. 235, show that the area of the enclosure is $2x(100 - x)$ sq. yd.

FIG. 235.

Draw a graph showing how the area alters for values of x from 20 to 80. For what values of x is the enclosed area 4000 sq. yd., 4500 sq. yd. ? What is the greatest area that can be enclosed ? Solve the equation $200x - 2x^2 = 3500$.

6. Draw the graph of $5x - 3x^2$ for values of x from -2 to 4. (i) Solve $5x - 3x^2 + 4 = 0$; (ii) solve $5x - 3x^2 = 1$; (iii) what is the greatest value of $5x - 3x^2$?

7. Draw the graph of $x + \frac{1}{x}$ for values of x from 0·2 to 4. (i) What is the smallest value of $x + \frac{1}{x}$, if x is positive ? (ii) Solve $x + \frac{1}{x} = 3{\cdot}5$. (iii) Solve $x^2 + 1 = 3x$.

Solve graphically, when possible, the following equations, Nos. 8-10 :

8. (i) $x^2 + 4x = 1$; (ii) $x^2 + 4x + 2 = 0$; (iii) $x^2 + 4x + 4 = 0$; (iv) $x^2 + 4x + 5 = 0$.

9. (i) $2x^2 - 3x = 3$; (ii) $2x^2 - 3x + 1 = 0$; (iii) $2x^2 - 3x + 2 = 0$; (iv) $10x^2 - 15x + 3 = 0$.

10. (i) $3x^2 - 5x = 2$; (ii) $3x^2 - 5x + 1 = 0$; (iii) $3x^2 - 5x + 2 = 0$; (iv) $3x^2 - 5x + 3 = 0$; (v) $6x^2 - 10x + 1 = 0$; (vi) $9x^2 - 15x = 4$.

Quadratic Equations.

The process of discovering a value (or values) of x, for which a given quadratic function of x is zero, is called, "solving a quadratic equation." We have seen that (i) in many cases no value of x, satisfying the equation, exists; (ii) if one value of x exists, there is in general a second value; (iii) it is possible to find the approximate values of x, if they exist, from an appropriate graph. We shall next proceed to explain methods for solving a quadratic equation by direct calculation.

If A and B are two expressions, and if all we know about them is that $A \cdot B = 2$, we have no means of finding the value of A by itself or of B by itself.

But if
$$A \cdot B = 0,$$
we must have *either* $A = 0$ *or* $B = 0$.

In other words, the product of two expressions cannot be zero unless one or other of them is zero. But if either of them is zero, their product must be zero.

Solution by Factors.

Example III. Solve: $x(x-2)=0$.

If $x=0$, the equation is satisfied, because
$$x(x-2)=0 \times (-2)=0.$$
If $x-2=0$ or $x=2$, the equation is satisfied, because $2 \times (0)=0$.

We put the argument as follows:
$$x(x-2)=0;$$
$$\therefore \textit{ either } x=0 \quad \textit{or} \quad x-2=0;$$
$$\therefore \mathbf{x=0} \quad \text{or} \quad \mathbf{x=2}.$$

Note. It is absurd to say, $x=0$ and 2, because x cannot equal 0 and 2 at the same time. The correct form of statement is $x=0$ or 2.

Example IV. Combine into a single statement
$$\text{Either } x=3 \quad \text{or} \quad x=-\tfrac{1}{2}.$$
$$\text{Either } x=3 \quad \text{or} \quad 2x=-1;$$
$$\therefore \text{ either } x-3=0 \quad \text{or} \quad 2x+1=0;$$
$$\therefore \text{ in each case, } (x-3)(2x+1)=0;$$
$$\therefore \mathbf{2x^2-5x-3=0}.$$

Exercise XIII. *d*.

1. If $xy=0$ and $x=1$, what must be the value of y?

2. If $xy=0$ and $x=0$, what do you know about the value of y?

3. If $xy=2$, can you say anything about the numerical value of x?

4. If $(a-b)x=0$, what can you say about the value of x?

5. If $(x-1)(y-2)=0$ and $x=10$, what do you know about y?

6. If $(x-3)(y-4)=0$, can you say anything about the value of x?

7. If $(x+1)(y+2)=0$ and if $x=-1$, can you say anything about y?

8. If $\dfrac{x+1}{y+2}=0$, can you say anything about the value (i) of x, (ii) of y?

9. What conclusion can you draw if $(x-11)(x+13)=0$? What do you get if you multiply out?

10. Combine into a single statement: "Either $x=2$ or $x=5$." Multiply out the result.

11. Combine into single statements the following and multiply out each result:

(i) Either $x=-2$ or $x=-5$; (ii) Either $x=-3$ or $x=4$;

(iii) $x=\pm 7$; (iv) Either $x=6$ or $x=0$;

(v) Either $x=0$ or $x=-8$;

(vi) Either $x=2$ or $x=5$ or $x=-3$.

12. What conclusion can you draw if $(6x-5)(11x+3)=0$? What do you get if you multiply out?

13. Combine into single statements, free from fractions, the following and multiply out each result:

(i) Either $x=\frac{1}{2}$ or $x=\frac{2}{3}$; (ii) Either $x=-\frac{1}{4}$ or $x=\frac{3}{5}$;

(iii) Either $x=-\frac{3}{4}$ or $x=-\frac{2}{7}$; (iv) Either $x=0$ or $x=-\frac{2}{5}$.

Solve the following equations, Nos. 14-34:

14. $(x-3)(x-7)=0$. 15. $(x+4)(x-5)=0$.

16. $(x+7)(x+2)=0$. 17. $x(x-10)=0$

18. $(x-3)^2=0$.
19. $(x+5)^2=0$.
20. $(x+1)(x+2)=0$.
21. $x(x+4)=0$.
22. $4x^2=0$.
23. $5(x-2)(x+8)=0$.
24. $(2x-1)(3x-5)=0$.
25. $(3x+1)(x+3)=0$.
26. $7(x-7)(4x-3)=0$.
27. $5(5x+2)(5x-3)=0$.
28. $6x(3x+7)=0$.
29. $(x-2)(x-5)(x+8)=0$.
30. $x(x+3)(x-6)=0$.
31. $x^2(x+4)=0$.
32. $2x(2x+1)(3x-10)=0$.
33. $x(x-4)^2=0$.
34. $(2x+5)(x-1)^2=0$.

Example V. Solve $\qquad x^2=9.$

First method: $\qquad x^2=9; \quad \therefore\ x^2-9=0;$

$$\therefore\ (x+3)(x-3)=0;$$

$$\therefore\ \text{either}\ x+3=0 \quad \text{or} \quad x-3=0;$$

$$\therefore\ x=-3 \quad \text{or} \quad x=3.$$

Second method: $\qquad x^2=9.$

Take the square root of each side: the square root of 9 is either $+3$ or -3, because $(+3)^2=9$ and $(-3)^2=9$.

$$\therefore\ x=+3 \quad \text{or} \quad -3, \text{ as before.}$$

Note. (i) $x=+3$ or -3 is usually written in the form $x=\pm 3$.

(ii) The second method is the shorter, but the pupil is very liable to forget that there are *two* square roots.

(iii) When taking the square root, we ought really to say

$$\pm x=\pm 3,$$

but this does not give anything more than is given by $x=\pm 3$.

Example VI. Solvo: $(x+3)(x+4)=2$.

$$(x+3)(x+4)=2; \quad \therefore\ x^2+7x+12=2;$$

$$\therefore\ x^2+7x+10=0;$$

$$\therefore\ (x+5)(x+2)=0;$$

$$\therefore\ \textit{either}\ x+5=0 \quad \text{or} \quad x+2=0;$$

$$\therefore\ x=-5 \quad \text{or} \quad x=-2.$$

Check: If $x=-5$, $(x+3)(x+4)=(-2)(-1)=2$.
If $x=-2$, $(x+3)(x+4)=(1)(2)=2$.

Exercise XIII. *e*.

Solve the following equations; check one answer by substitution.

1. $x^2-7x+12=0$.
2. $x^2+7x+12=0$.
3. $x^2-3x+2=0$.
4. $x^2+3x+2=0$.
5. $x^2-x-20=0$.
6. $x^2-8x+12=0$.
7. $x^2+2x=15$.
8. $x^2-8x+15=0$.
9. $x^2+11x+24=0$.
10. $x^2+11x+30=0$.
11. $x^2+x=72$.
12. $x^2-13x+22=0$.
13. $x^2-5x=0$.
14. $x^2=3x$.
15. $2x^2=18$.
16. $4x^2=25$.
17. $2x^2+3x=0$.
18. $5x^2=7x$.
19. $3-x^2=x^2-5$.
20. $(3x)^2=100$.
21. $3x^2=11x$.
22. $2x^2-15x+25=0$.
23. $2x^2+5x=7$.
24. $(x-2)^2=4$.
25. $4x^2+4x+1=0$.
26. $2(x^2+6)=25x$.
27. $3x^2=2x+1$.
28. $5x^2+2=11x$.
29. $6x^2+5x-6=0$.
30. $10x^2-30x+20=0$.
31. $3x^2+15x+12=0$.
32. $x^2+\frac{1}{4}=x$.
33. $x+1=\dfrac{6}{x}$.
34. $\dfrac{x}{3}-\dfrac{3}{x}=\dfrac{3}{2}$.
35. $x^3=9x$.

Example VII. Find the equation whose roots are a, b.

We can say that $\quad x=a \quad$ or $\quad x=b$

$$\text{if} \quad x-a=0 \quad \text{or} \quad x-b=0,$$

$$\text{or if} \quad (x-a)(x-b)=0;$$

$\therefore$ the required equation is

$$x^2-ax-bx+ab=0,$$

$$\text{or} \quad \mathbf{x^2-(a+b)x+ab=0}.$$

From this result we can draw a most important conclusion. If we have a quadratic equation in which the coefficient of x^2 is 1,

(i) the sum of the roots, $a+b$, is equal to "minus the coefficient of x";

(ii) the product of the roots, ab, is equal to "the constant term."

This gives us a valuable method of checking our answers.

For example, if we find that the roots of $2x^2+3x-14=0$ are 2 and $-3\frac{1}{2}$, we can check these results as follows:

Divide the equation by 2 to make the coefficient of x^2 unity.

$$\therefore\ x^2+\tfrac{3}{2}x-7=0.$$

Then the sum of the roots should be $-\frac{3}{2}$, and the product of the roots should be -7.

Check:
$$2+(-3\tfrac{1}{2})=-1\tfrac{1}{2}=-\tfrac{3}{2},$$
$$2\times(-3\tfrac{1}{2})=-2\times\tfrac{7}{2}=-7.$$

Exercise XIII. *f*.

Solve the following equations, Nos. 1-18; check your answers by finding the sum and product of the roots.

1. $x^2-9x+14=0.$
2. $x^2-5x-14=0.$
3. $x^2+3x=70.$
4. $x^2+11x+28=0.$
5. $10x^2-11x+3=0.$
6. $3x^2-8x=3.$
7. $4y+21=y^2.$
8. $12z=9z^2+4.$
9. $6p=(p+4)(p-4).$
10. $\dfrac{n(n+1)}{2}=28.$
11. $64t-16t^2=48.$
12. $y^2+(6-y)^2=20.$
13. $(x+3)^2+(x-3)^2=20x.$
14. $\dfrac{n}{2}[6+2(n-1)]=120.$
15. $\dfrac{24}{x}+\dfrac{24}{x-5}=11.$
16. $\dfrac{210}{x+2}+\dfrac{210}{x+4}=72.$
17. $\left(\dfrac{60}{x}+5\right)(x-6)=60.$
18. $\dfrac{x}{x-2}+\dfrac{x-2}{x}=3\tfrac{1}{3}.$

Write down the sum and product of the roots of the equations in Nos. 19-26; check by solving.

19. $x^2-5x+6=0.$
20. $x^2-5x-24=0.$
21. $x^2+5x+4=0.$
22. $x^2+5x-6=0.$
23. $8x^2-6x+1=0.$
24. $6x^2-x-2=0.$
25. $50x^2+25x+2=0.$
26. $50x^2+25x-3=0.$

EXTRA PRACTICE EXERCISES. E.P. 12.

Quadratic Equations.

1. If $(x-3)(y-5)=0$, can you find the value of x when (i) $y=7$; (ii) $y=5$; (iii) $y=0$? If so, what is it?

2. If $(a-b)(x+2)=0$, find, when possible, the value of x when (i) $a=3$, $b=1$; (ii) $a=0$, $b=1$; (iii) $a=4$, $b=4$?

3. Do you know anything about the numerical value of x if $xy=10$?

4. What can you say about the values of x and y if

$$(x-3)(y-5)=0?$$

Express the following facts by equations:

5. Either $x=3$ or $x=6$.
6. Either $y=2$ or $y=5$.
7. Either $t=2$ or $t=-2$.
8. Either $z=0$ or $z=4$.
9. Either $x=1$ or $y=1$.
10. Either $x=-2$ or $x=3$.
11. Either $p=-4$ or $p=-7$.
12. Either $y=3$ or $z=-2$.
13. Either $x=\frac{1}{2}$ or $x=3$.
14. Either $t=\frac{2}{3}$ or $t=\frac{4}{7}$.
15. Either $y=0$ or $y=-\frac{3}{4}$
16. Either $z=-2\frac{1}{2}$ or $z=-1\frac{1}{3}$.
17. $x=\pm 5$.
18. $s=\pm\frac{2}{3}$.
19. $y=0$ or 2 or 5.
20. $z=1$ or -1 or -2.
21. $t=\frac{1}{2}$ or -2 or $1\frac{1}{4}$.
22. $v=0$ or $\pm\frac{3}{4}$.
23. $x=-\frac{2}{5}$ or $2\frac{1}{2}$.
24. $y=1$ or $-1\frac{1}{2}$ or $-\frac{1}{3}$.

Solve the following equations:

25. $x(x-3)=0$.
26. $(x-2)(x+3)=0$.
27. $(x-4)^2=0$.
28. $3x(x+2)=0$.
29. $(2x-3)(x-5)=0$.
30. $(3x+1)(x+3)=0$.
31. $5x(5x-2)=0$.
32. $(2x+3)^2=0$.
33. $x(x+1)(x-2)=0$.
34. $(2x-7)(7x+2)=0$.
35. $(x-4)^2(x+4)=0$.
36. $(3-2x)(5+x)=0$.
37. $(4+x)(1-4x)=0$.
38. $(1-x)(2-x)(3+x)=0$.
39. $x^2-8x+15=0$.
40. $x^2-8x=0$.

41. $x^2+11x+28=0$.
42. $x^2+11x=0$.
43. $x^2+5x-6=0$.
44. $x^2-3x-70=0$.
45. $x^2-6x+9=0$.
46. $x^2+12x+36=0$.
47. $x^2+9x=36$.
48. $x^2-11x=60$.
49. $x(x-1)=72$.
50. $x(x+2)=99$.
51. $2x^2-11x+5=0$.
52. $6x^2=x+2$.
53. $x^2=25$.
54. $4x^2=9$.
55. $15x^2+2x=8$.
56. $14x^2=17x+6$.
57. $(2x+3)^2=25$.
58. $20x^2-7x=6$.
59. $4x^2=7x$.
60. $10x^2+33x+20=0$.
61. $(x+1)(x-3)=12$.
62. $(2x-1)(1+3x)=4$.
63. $x(x-1)=3(x-1)$.
64. $(x+2)^2=5(x+2)$.
65. $x-\frac{18}{x}=7$.
66. $1+\frac{8}{x-1}=\frac{18}{x+1}$.
67. $\frac{10}{2x-1}=3\left(1-\frac{1}{x}\right)$.
68. $x+4=\frac{9}{x+4}$.
69. $\frac{1}{x-1}-\frac{1}{x+2}=\frac{1}{6}$.
70. $\frac{x-1}{2x+1}=\frac{4}{3x}$.

SUPPLEMENTARY EXERCISE. S. 9

1. If the square of $2x+1$ equals the square of $2x+7$, what is the value of x?

2. If $y+\frac{3}{x}=x+\frac{3}{y}$, find y when $x=1$.

Find x from the following equations, Nos. 3-13:

3. $\frac{x}{2}=\frac{18}{x}$.
4. $5\sqrt{x}=45$.
5. $\frac{7}{\sqrt{x}}=\frac{21}{2}$.
6. $x^2-ax=0$.
7. $x^2-a^2=0$.
8. $4x^2=9c^2$.
9. $m^2x^2=n^2$.
10. $ax^2+bx=0$.
11. $x^2=px+qx$.
12. $x^2+px-6p^2=0$.
13. $3x^2-11bx+10b^2=0$.

14. Show that $x=1$ satisfies $x^2+17x-18=0$. What is the sum of the roots? What is the second root?

15. One of the roots of $x^2 - 7x + 4 = 0$ is $x = 6{\cdot}37$ correct to two decimal places. Assuming this fact, *write down* the other root.

16. If $x = 3$ is one root of $x^2 + bx - 15 = 0$, what is the other root? Then find b.

17. If $x = -2$ is one root of $x^2 - 17x + c = 0$, what is the other root? Then find c.

18. If $x = 2$ is one root of $3x^2 - 5x + c = 0$, what is the other root? Then find c.

19. Form the equations whose roots are (i) $2c$ and $-3c$; (ii) $a + b$ and $a - b$.

20. Find x if $(x + 13)^2 - 3(x + 13) - 10 = 0$.

21. Can you find the value of $\frac{x}{y}$ if $x^2 - 6xy + 8y^2 = 0$, given that $y \neq 0$?

22. What can you say about the values of a and b, if

$$(a - 1)(a - 2) = (b - 1)(b - 2),$$

when a is not equal to b?

23. Find x and y if $y = x(x - 2)$ and $\frac{1}{3}y = x + 2$.

24. Find x and y if $y = x + 4$ and $xy = 21$.

25. Find x and R, given that $Rx = 12$ and $R\sqrt{(1 - x^2)} = 5$.

EASY REVISION PAPERS. A. 31-35

A. 31

1. What are the factors of (i) $4a^2 + 4ab$; (ii) $6ab + 6b^2$? What is their L.C.M.?

Simplify

$$\text{(i) } \frac{4a^2 + 4ab}{6ab + 6b^2}; \quad \text{(ii) } \frac{1}{4a^2 + 4ab} + \frac{1}{6ab + 6b^2}.$$

2. Solve $$\frac{x + 3}{y + 5} = \tfrac{1}{2}, \quad \frac{x - 1}{y - 1} = \tfrac{1}{3}.$$

3. Express as a compound quantity:
(i) (xs. yd.) $\times 2$ if $6 < y < 12$; (ii) (£a bs.) $\times 2$ if $10 < b < 20$.

4. For what values of x is $(3x + 1)(2x - 5)$ equal to 0?

5. Make l the subject of the formula $S = \pi r(l + r)$.

A. 32

1. Give a general statement which includes the following:

3×5 is less than the square of 4 by 1;
10×12 is less than the square of 11 by 1;
37×39 is less than the square of 38 by 1.

Prove that your statement is true.

2. Factorise (i) $10x^2 - 10$; (ii) $2x^2 - 4x + 2$.
Simplify (i) $(10x^2 - 10) \div 5(x+1)$; (ii) $(2x^2 - 4x + 2) \div (2x - 2)$.

3. By means of an equation, divide a hundredweight into two parts, so that one is three-fifths of the other. Ans. in cwt.

4. Find a number which when divided by $p+1$ gives the same result as when $p-1$ is divided by p.

5. Solve $$(x+7)(x-1) = 273.$$
Use your answer to write down two numbers which differ by 8 and whose product is 273.

A. 33

1. A discount of 1s. in the £ is taken off a bill of £a. What is the net amount in £?

2. Solve (i) $ax + a = bx + b$;

(ii) $\dfrac{2x}{3} - y = 4\frac{1}{2}, \quad \dfrac{5x}{6} + y = 9.$

3. Factorise $12x^2 + 5x - 2$. What values of x make $12x^2 + 5x - 2$ (i) equal to 0, (ii) equal to -2?

4. Factorise (i) $a^9 - a^8$; (ii) $a^4 - a^3$.

Simplify $$\frac{a^4 - a^3}{a^9 - a^8}.$$

5. In a factory 340 people are employed. The men are paid 4s. 6d. a day and the women are paid 3s. a day; the total wages amount to £60 a day. How many men are employed?

A. 34

1. Make n the subject of the formula: $I = \dfrac{E}{\dfrac{r}{n} + R}$.

2. The cost of sending x gallons of milk 40 miles is $(a + bx)$ pence, where a, b are the same for all quantities. The cost for 10 gallons is 1s. $3\frac{1}{2}$d. and for 60 gallons is 6s. 6d. Find the cost for x gallons.

3. Simplify (i) $\dfrac{x^3-x}{x^3-x^2}$; (ii) $\dfrac{1}{x^3-x}-\dfrac{1}{x^3-x^2}$.

4. Show that $x=2$ makes $x^3-x^2+7x-18$ equal to 0. Write down one factor of $x^3-x^2+7x-18$ and then find another factor.

5. A picture frame is 22 in. long and 16 in. wide. The width of the frame is x in. Find the surface area of the frame in terms of x. Find x if this area is 105 sq. in.

A. 35

1. Write down the square of $5a-3x$.
Add a term to $4x^2-28xy$ so as to make it a perfect square. Express $4x^2-28xy$ as the difference of two squares.

2. (i) Simplify $\dfrac{3}{x-4}+\dfrac{4}{x-3}-\dfrac{7x}{x^2-7x+12}$.

(ii) Solve $\dfrac{x+a-b}{x+a+b}=\dfrac{a+b}{a-b}$.

3. Interpret the formulae $C=\pi d$ and $A=\frac{1}{4}\pi d^2$. Then find A in terms of C, π only.

4. What is the remainder if the product of $2x+5$ and $x-1$ is divided by $x+2$?

5. Find a number such that, whether divided into two equal parts or into three equal parts, the product of the parts shall be the same.

CHAPTER XIV

THE GENERAL QUADRATIC

Solution by Completing the Square.

THE process of adding a term to $x^2 + 6x$ to make it a perfect square has been explained on p. 264.

Example I. Solve $\quad x^2 - 6x = 27.$

Now $\quad x^2 - 6x + 3^2 = (x-3)^2.$

The given equation may therefore be written

$$x^2 - 6x + 3^2 = 27 + 9 \quad \text{or} \quad (x-3)^2 = 36.$$

Take the square root of each side;

$$\therefore\ x - 3 = \pm 6\ ;$$
$$\therefore\ x = 6 + 3 \quad \text{or} \quad x = -6 + 3\ ;$$
$$\therefore\ x = 9 \quad \text{or} \quad x = -3.$$

Note. (i) When taking a square root, *remember* there are two answers.

(ii) In this case it would have been far quicker to use the previous method, thus:

$$x^2 - 6x - 27 = 0\ ; \quad \therefore\ (x-9)(x+3) = 0\ ;$$
$$\therefore\ x - 9 = 0 \quad \text{or} \quad x + 3 = 0\ ;$$
$$\therefore\ x = 9 \quad \text{or} \quad x = -3.$$

The method of "completing the square" should only be used when no simple factors can be found, as in the next two examples.

Example II. Solve $x^2 - 5x + 2 = 0$, giving the roots of the equation correct to two places of decimals.

$$x^2 - 5x = -2\ ;$$
$$\therefore\ x^2 - 5x + \left(\frac{5}{2}\right)^2 = -2 + \frac{25}{4}\ ;$$
$$\therefore\ \left(x - \frac{5}{2}\right)^2 = \frac{17}{4}.$$

Take the square root of each side ;

$$\therefore\ x-\frac{5}{2}=\pm\frac{\sqrt{17}}{2};$$

$$\therefore\ x=\frac{5}{2}+\frac{\sqrt{17}}{2}\quad\text{or}\quad x=\frac{5}{2}-\frac{\sqrt{17}}{2}.$$

But $\sqrt{17}\simeq 4{\cdot}12$;

$$\therefore\ x=2{\cdot}5+2{\cdot}06\quad\text{or}\quad x=2{\cdot}5-2{\cdot}06;$$

$$\therefore\ x=4{\cdot}56\quad\text{or}\quad x=0{\cdot}44.$$

Check : The sum of the roots is $4{\cdot}56+0{\cdot}44=5$.

The product of the roots is $\simeq 4{\cdot}6\times 0{\cdot}44\simeq 2$.

Note. The square root of 17 can be found by rule to as many places of decimals as is required, but it is impossible to find any whole number or fraction whose square is exactly 17. The square root of 17 is therefore called an *irrational number* ; but each square root of, say, $6\frac{1}{4}$ or $\frac{25}{4}$ is a *rational number*, viz. $\pm\frac{5}{2}$ or $\pm 2\frac{1}{2}$, because $\frac{5}{2}\times\frac{5}{2}$ is exactly equal to $\frac{25}{4}$.

We may use the factor-method, even when the roots are not rational, as follows :

Starting from $\left(x-\frac{5}{2}\right)^2=\frac{17}{4}$, obtained above, we have

$$\left(x-\frac{5}{2}\right)^2-\frac{17}{4}=0;$$

$$\therefore\ \left(x-\frac{5}{2}\right)^2-\left(\frac{\sqrt{17}}{2}\right)^2=0;$$

$$\therefore\ \left[\left(x-\frac{5}{2}\right)+\frac{\sqrt{17}}{2}\right]\left[\left(x-\frac{5}{2}\right)-\frac{\sqrt{17}}{2}\right]=0;$$

$$\therefore\ \text{either } x-\frac{5}{2}+\frac{\sqrt{17}}{2}=0\quad\text{or}\quad x-\frac{5}{2}-\frac{\sqrt{17}}{2}=0;$$

$$\therefore\ \text{either } x=\frac{5}{2}-\frac{\sqrt{17}}{2}\quad\text{or}\quad x=\frac{5}{2}+\frac{\sqrt{17}}{2},\text{ as before.}$$

This method is longer, but it shows clearly why there must be two roots.

Example III. Solve $3x^2+11x+4=0$, giving the roots correct to two places of decimals.

$$3x^2+11x=-4;\quad\therefore\ x^2+\frac{11}{3}x=-\frac{4}{3}.$$

$$\left[\text{Now}\quad x^2+\frac{11}{3}x+\left(\frac{11}{6}\right)^2=\left(x+\frac{11}{6}\right)^2\right];$$

$$\therefore\ x^2+\frac{11}{3}x+\left(\frac{11}{6}\right)^2=\frac{121}{36}-\frac{4}{3};$$

$$\therefore\ \left(x+\frac{11}{6}\right)^2=\frac{121-48}{36}=\frac{73}{36}.$$

Take the square root of each side.

$$\therefore\ x+\frac{11}{6}=\pm\frac{\sqrt{73}}{6};$$

$$\therefore\ x=-\frac{11}{6}+\frac{\sqrt{73}}{6}\quad\text{or}\quad x=-\frac{11}{6}-\frac{\sqrt{73}}{6}.$$

$$\therefore\ x=\frac{-11+\sqrt{73}}{6}\quad\text{or}\quad x=\frac{-11-\sqrt{73}}{6}.$$

Now $\sqrt{73}\simeq 8{\cdot}54$;

$$\therefore\ x\simeq\frac{-2{\cdot}46}{6}\quad\text{or}\quad\frac{-19{\cdot}54}{6};$$

$$\therefore\ x\simeq -0{\cdot}41\quad\text{or}\quad -3{\cdot}26.$$

Check: Sum of roots $\simeq -0{\cdot}41-3{\cdot}26=-3{\cdot}67$.

$$-(\text{coefficient of } x)=-\frac{11}{3}=-3\tfrac{2}{3}\simeq -3{\cdot}67.$$

Product of roots $\simeq -0{\cdot}4\times -3{\cdot}3\simeq 1{\cdot}3$.

Constant term $=\frac{4}{3}\simeq 1{\cdot}3$.

Note. The results, $x=\dfrac{-11+\sqrt{73}}{6}$ or $\dfrac{-11-\sqrt{73}}{6}$, are usually written more shortly in the form

$$x=\frac{-11\pm\sqrt{73}}{6}.$$

Exercise XIV. *a*.

What number must be added to the expressions in Nos. 1-9 to make the result a perfect square? Of what is it then the square?

1. x^2-6x.
2. x^2+10x.
3. x^2+3x.
4. x^2-7x.
5. $x^2+\frac{10}{3}x$.
6. $x^2-\frac{4}{7}x$.
7. $x^2-\frac{3}{5}x$.
8. $x^2+\frac{11}{2}x$.
9. $x^2-\frac{7}{9}x$.

The equations in Nos. 10-15 have rational roots. Solve them first by completing the square; then solve by the direct

factor method. Compare the length of the work and the results in the two cases.

10. $x^2-6x=40.$

11. $x^2+5x=14.$

12. $x^2-10x+21=0.$

13. $x^2+13x+22=0.$

14. $x^2+x=12.$

15. $x^2-9x=52.$

Solve the following equations, Nos. 16-55.

Use the direct factor method whenever it is easier to do so. If the roots are not rational, work out each root correct to two places of decimals.

16. $x^2=2.$

17. $3x^2=1$

18. $2x^2=25.$

19. $\frac{22r^2}{7}=10.$

20. $x^2+9=36.$

21. $2(x^2+1)=15.$

22. $x^2+4x-2=0.$

23. $x^2-4x-2=0.$

24. $x^2+2x-1=0.$

25. $x^2-10x+25=0.$

26. $x^2+12x=5.$

27. $x^2-20x=33.$

28. $x^2+5x=3.$

29. $x^2+5x+2=0.$

30. $x^2-x=4.$

31. $x^2-7x+9=0.$

32. $x^2-7x=30.$

33. $x(x+1)=8.$

34. $3x^2+4x=2.$

35. $5x^2-8x+2=0.$

36. $2x^2+12x=7.$

37. $3x^2+5x=2.$

38. $3x^2=8x-2.$

39. $7x^2-4x=4.$

40. $2x^2+7x=5.$

41. $x^2+20=11x.$

42. $4x^2+3x-4=0.$

43. $x^2+x=120.$

44. $6x^2-7x=3.$

45. $10x^2=x+1.$

46. $x(2x+9)=10.$

47. $30x^2-20x-15=0.$

48. $\frac{2x^2}{3}=x+2.$

49. $R(R+5)=84.$

50. $(R+5)(R-1)=16.$

51. $(t-2)(t-7)=36.$

52. $\frac{1}{x}+\frac{1}{x+1}=\frac{1}{2}.$

53. $\frac{x}{x+2}=\frac{x+1}{2x-5}.$

54. $\frac{2x}{x+3}+\frac{3}{x+4}=1.$

55. $\frac{x}{x+1}+\frac{2x}{x+2}=2.$

Solution by Formula.

All quadratic equations can be reduced to the form

$$Ax^2+Bx+C=0.$$

Therefore, if we can solve this equation, we can write down the answer for any other quadratic equation, by giving A, B and C particular values.

Solution of $Ax^2+Bx+C=0$.

Divide throughout by A; $\therefore\ x^2+\frac{B}{A}x+\frac{C}{A}=0$;

$$\therefore\ x^2+\frac{B}{A}x+\left(\frac{B}{2A}\right)^2=-\frac{C}{A}+\frac{B^2}{4A^2};$$

$$\therefore\ \left(x+\frac{B}{2A}\right)^2=\frac{B^2-4AC}{4A^2};$$

$$\therefore\ \left(x+\frac{B}{2A}\right)^2=\left(\frac{\sqrt{B^2-4AC}}{2A}\right)^2.$$

Take the square root of each side;

$$\therefore\ x+\frac{B}{2A}=\pm\frac{\sqrt{B^2-4AC}}{2A};$$

$$\therefore\ x=-\frac{B}{2A}\pm\frac{\sqrt{B^2-4AC}}{2A};$$

$$\therefore\ x=\frac{-\mathbf{B}\pm\sqrt{\mathbf{B}^2-4\mathbf{AC}}}{2\mathbf{A}}.$$

Note. The above answer is an abbreviation for

$$x=\frac{-B+\sqrt{B^2-4AC}}{2A}\quad\text{or}\quad x=\frac{-B-\sqrt{B^2-4AC}}{2A}.$$

It is evident that the sum of these roots is $\frac{-2B}{2A}=-\frac{B}{A}$. We could also show by direct multiplication that their product is $\frac{C}{A}$.

Example IV. Solve $4x^2-1=7x$ and express the answer correct to two places of decimals.

The equation may be written $4x^2-7x-1=0$.

$\therefore$ the equation $Ax^2+Bx+C=0$ is equivalent to the given equation if $A=4$, $B=-7$, $C=-1$.

Substitute in $x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$.

Then $$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(4)(-1)}}{8}$$

$$=\frac{7\pm\sqrt{49+16}}{8}=\frac{7\pm\sqrt{65}}{8}.$$

But $\sqrt{65}\simeq 8{\cdot}06$;

$$\therefore\ x\simeq\frac{7\pm 8{\cdot}06}{8}\simeq\frac{15{\cdot}06}{8}\quad\text{or}\quad-\frac{1{\cdot}06}{8}$$

$$\simeq 1{\cdot}88\quad\text{or}\quad -0{\cdot}13.$$

Note. (i) Substitute first and simplify afterwards; do not try to do both at once.

(ii) Do not use the formula if you can find rational factors; the direct factor method, when possible, is the quicker.

Exercise XIV. *b*.

Solve the following equations: use the direct factor method whenever the roots are rational. If the roots are not rational, use the formula and work out each root correct to two places of decimals.

1. $2x^2 + 4x + 1 = 0.$
2. $2x^2 - 8x + 7 = 0.$
3. $2x^2 + 5x + 3 = 0.$
4. $x^2 + x = 1.$
5. $3x^2 + 4x = 4.$
6. $x^2 - 12x + 25 = 0.$
7. $x^2 - x = 2.$
8. $3x^2 - 5 = 3x.$
9. $x^2 + x = 40.$
10. $12x^2 - 6x = 1.$
11. $5x^2 = 2x + 7.$
12. $3x + 1 = 5x^2.$
13. $2x^2 + 6x + 3 = 0.$
14. $2x^2 - 7x + 6 = 0.$
15. $\frac{x^2}{3} - \frac{x}{2} = \frac{1}{5}.$
16. $x + \frac{\text{[illegible]}}{x} = 11.$
17. $\frac{x}{x-2} + \frac{x-2}{x} = 4.$
18. $\frac{x+1}{x-1} + \frac{3-x}{x+3} = 1.$

EXTRA PRACTICE EXERCISES. E.P. 13.

The General Quadratic.

Solve the following equations. If the roots are not rational, give each root correct to one place of decimals.

1. $x^2 - 4x = 3.$
2. $x^2 + 6x = 8.$
3. $x^2 - 5x = 7.$
4. $x^2 + 3x = 5.$
5. $x^2 + 8x + 14 = 0.$
6. $x^2 - 10x + 12 = 0.$
7. $x^2 - 7x = 18.$
8. $x^2 + 5x - 7 = 0.$
9. $x^2 + x - 3 = 0.$
10. $2x^2 - 12x = 15.$
11. $2x^2 + 3x = 4.$
12. $3x^2 - 2x = 1.$
13. $3x^2 - 2x = 2.$
14. $5x^2 - 8x + 2 = 0.$
15. $4x^2 - 3x = 3.$
16. $x + \frac{1}{x} = 3.$
17. $x + 5 = \frac{2}{x}.$
18. $2x + \frac{1}{x} = 7.$
19. $\frac{1}{x+1} + \frac{1}{x-1} = \frac{4}{5}.$
20. $\frac{x}{x-1} + 1 = \frac{x-2}{2x}.$
21. $\frac{x}{x-2} - \frac{x+1}{x+4} = 1.$
22. $\frac{24}{x+4} + \frac{9}{x-5} = 5.$

23. $\frac{2x}{x-1}-\frac{3}{x+2}=3.$

24. $\frac{x}{x+2}+\frac{x+2}{x}=5.$

25. $\frac{20}{x+1}+\frac{24}{x+2}=8.$

26. $\frac{3}{x+5}-\frac{1}{2x+3}=\frac{1}{4}.$

27. $\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}=0.$

28. $\frac{30}{x+1}+\frac{36}{x-1}+6=0.$

29. $\frac{1}{x+1}+\frac{2}{x-1}=\frac{1}{x+3}.$

30. $\frac{3}{x+1}+\frac{2}{x-1}=\frac{3(x-1)}{2(x+1)}.$

If more practice in the use of the formula is required, Exercise XIV. *a*, Nos. 22-48, may be used.

Problems.

Example V. The distance from London to Carlisle is 300 miles. One train goes 10 miles an hour faster than another and takes $1\frac{1}{2}$ hours less on the journey. Find the speed of each.

Let the speed of the first train be x miles an hour.

Then the speed of the second train is $(x-10)$ miles an hour.

$\therefore$ the first train takes $\frac{300}{x}$ hours and the second train takes $\frac{300}{x-10}$ hours for the journey.

But the time of the second train is $1\frac{1}{2}$ hours more than the time of the first train.

$$\therefore \frac{300}{x-10}=\frac{300}{x}+\frac{3}{2};$$

$$\therefore 600x=600(x-10)+3x(x-10);$$

$$\therefore 600x=600x-6000+3x^2-30x;$$

$$\therefore 3x^2-30x-6000=0;$$

$$\therefore x^2-10x-2000=0;$$

$$\therefore (x-50)(x+40)=0;$$

$$\therefore x-50=0 \quad \text{or} \quad x+40=0;$$

$$\therefore x=50 \quad \text{or} \quad -40.$$

A negative value of x is clearly unsuitable.

$\therefore$ we take $x=50$ and disregard $x=-40$.

Also, if $x=50$, $x-10=40$.

$\therefore$ the speeds of the two trains are 50 miles per hour and 40 miles per hour.

Exercise XIV. c.

1. The square of a number is equal to seven times that number. Find the number.

2. I think of a number; then square it and add the original number; the result is 56. What number did I choose?

3. Find three consecutive integers, such that the square of one is equal to the sum of the squares of the other two.

4. The hypotenuse of a right-angled triangle is twice one of the other sides, and the third side is 3 inches; find the length of the hypotenuse, correct to two significant figures.

5. The length of a rectangle is 2 ft. more than its width, and the area is 48 sq. ft.; find its length.

6. In Fig. 236, $AC \cdot CB$ is 12 sq. cm., find AC

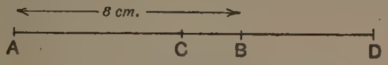

Fig. 236.

7. In Fig. 236, $AD \cdot BD$ is 33 sq. cm., find AD.

8. The sum of the first n integers 1, 2, 3, 4, 5, ... is $\frac{1}{2}n(n+1)$. How many must be added together to make up 91?

9. A stone is thrown up in such a way that after t seconds its height above the ground is $(80t - 16t^2)$ feet. When will it strike the ground again?

10. With the data of No. 9, find after what time the stone is 75 feet above the ground.

11. From A to B is 60 miles. If a man motors instead of cycling, he goes 8 miles an hour faster and saves 2 hours. How fast does he cycle?

12. An n-sided figure has $\frac{1}{2}n(n-3)$ diagonals. How many sides has a figure, if it has 135 diagonals?

13. From London to Crewe is 160 miles. If the speed of an express train is decreased by 2 miles an hour, the time of the journey is lengthened by 8 minutes. Find the new time taken.

14. Fig. 237 represents a rectangular lawn with a rectangular flower-bed in the middle; the grass is x yards wide on each

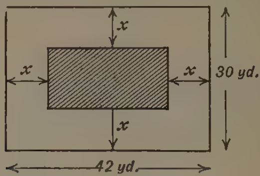

FIG. 237.

side and the grass-area is 720 sq. yd. Find the dimensions of the flower bed.

15. The seat of a tramcar is $16\frac{1}{2}$ ft. long. If it holds one person more than the proper number, each person has $1\frac{1}{2}$ inches less width than he should. How many is it supposed to hold?

16. A flat circular ring, *i.e.* the space between two concentric circles, has an area of $16\frac{1}{2}$ sq. in.; the radius of the inner circle is 5 inches. What is the width of the ring? [Take $\pi = \frac{22}{7}$; area of circle $= \pi r^2$.]

17. A man spends £12 a year on coal. The price goes up 6s. a ton and he then reckons he must use 2 tons less every year, if he is to spend the same amount as previously. How much coal did he use formerly?

18. The sum of the ages of a class of boys is 252 years. If two boys, each 9 years old, join the class, the average age is reduced by 6 months. Find the original number in the class.

19. The distance that a ball rolls down a sloping groove in t seconds is $(10t + 2t^2)$ inches. How long will it take to roll a distance of 11 feet?

20. According to an architect's plan, a room has an area of 400 sq. ft. The plan is altered so that the length is made 5 ft. less and the width 4 ft. more, without altering the area. Find the new length and breadth.

21. The corners of Fig. 238 are all right-angled, and the measurements are in inches; find x, if the area of the figure is $\frac{7}{8}$ sq. foot.

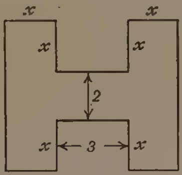

FIG. 238.

22. A man rows 5 miles an hour in still water. He rows 7 miles up a river and back again in $3\frac{1}{3}$ hours. How fast does the stream flow?

23. Find the distance from B of a point P on BC in Fig. 239 such that $PA = PD$. [The point P is not shown in the figure.]

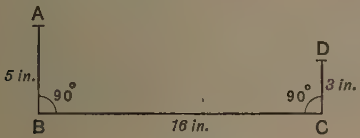

FIG. 239.

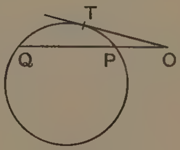

FIG. 240.

24. If OT is a tangent to the circle in Fig. 240, it can be proved that $OT^2 = OP \cdot OQ$; find OQ, given that

(i) $PQ = 9$ in., $OT = 6$ in.;

(ii) $PQ = 8$ in., $OT = 6$ in.

25. If $\frac{1}{2}(x-1)(x-2)$ and $\frac{1}{5}(x+1)(x+3)$ are consecutive integers, what integers are they?

SUPPLEMENTARY EXERCISE. S. 10

1. a, b, c, d are four consecutive integers such that the sum of the squares of a and b exceeds the square of d by 4. Find c.

2. Find three consecutive integers such that their product is 40 times their sum.

3. In Fig. 241, if $c=25$ and $a+b=31$, find a.

4. In Fig. 241, if $c=4$ and $b-a=1$, find b correct to two places of decimals.

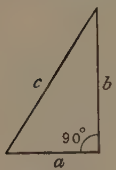

FIG. 241.

5. Divide 1 into two parts, so that the sum of their cubes is $\frac{1}{3}$.

6. The hot-water tap takes 8 minutes longer than the cold-water tap to fill a bath ; both taps together take 3 minutes. How long does the hot-water tap take by itself ?

7. A stone is projected upwards in such a way that after t seconds it is at a height of $(120t-16t^2)$ feet. How long is it in the air ? If it takes the same time to go up as come down, what is the greatest height it reaches ?

8. The sum of the first n numbers of the set 3, 7, 11, 15, 19, ... is $n(2n+1)$. How many must be taken to add up to 210 ? What will then be the last number taken ?

9. A Zeppelin starts from a base 250 miles from London and can travel at 60 miles an hour for 10 hours in still air. What rate of wind will prevent it from returning to its base after a raid on London, if the wind blows continuously from the base to London ?

10. In Fig. 242, $AB=4$ in., $BC=5$ in., $CD=6$ in. If $PD=2PA$, find PB.

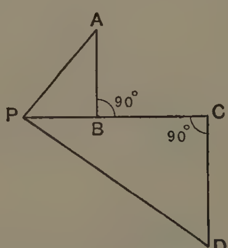

FIG. 242.

11. A sphere is coated with material 1 inch thick ; the volume of this covering is 400 cu. inches ; find the radius of the sphere correct to $\frac{1}{10}$th of an inch. [The volume of a sphere of radius r in. is $\frac{4}{3}\pi r^3$ cu. in.]

12. Find a whole number such that the sum of its square and its cube is sixteen times the next greater whole number.

13. A ladder 26 ft. long rests against a wall with its foot at a distance of 10 ft. from the wall. How far must the lower end be pulled out from the wall so that the top moves down through an equal distance?

14. A man has to travel 50 miles in 4 hours; he does it by walking the first 7 miles at x miles an hour, bicycling the next 7 miles at $4x$ miles an hour and motoring the remainder at $(6x+3)$ miles an hour. Find x.

15. A telegraph wire, of length l feet, connects the tops A, B of two poles, x feet apart (see Fig. 243); if the sag EF at the middle is k feet, it is known that $k=\sqrt{\left\{\frac{3l(l-x)}{8}\right\}}$. Find in terms of k, l the excess of the arc AB over the chord AB. Find also the length of wire required to join two poles 30 yd. apart, allowing for a maximum sag of 2 feet.

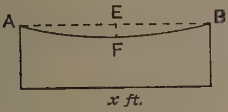

FIG. 243.

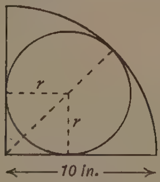

FIG. 244.

16. Fig. 244 represents a circle of radius r inches, inscribed in a quadrant of radius 10 inches. Prove that $2r^2=(10-r)^2$, and find r correct to two places of decimals.

17. The area of the curved surface of a cone whose base-radius is r inches and whose height is h inches is $\pi r\sqrt{(r^2+h^2)}$ sq. inches. If the height of a cone is 3 inches, and if the area of its curved surface is equal to the area of a circle of radius 4 inches, find the radius of its base, correct to $\frac{1}{10}$ inch.

18. If the graphs of $2x^2-3$ and $3x+2$ are drawn with the same scale and axes, calculate the values of x corresponding to their points of intersection. Sketch *roughly* the graphs.

CHAPTER XV

FORMULAE AND LITERAL EQUATIONS

Change of Subject.

SOME examples of changing the subject of a formula have been given in Chapter IV. We shall now proceed to consider some rather more difficult questions of the same kind.

Example I. The volume, V cu. in., of a circular cone of base-radius r in. and height h in. is given by the formula

$$V=\tfrac{1}{3}\pi r^2 h.$$

Make (i) h, (ii) r the subject of the formula.

(i) $$\tfrac{1}{3}\pi r^2 h=V; \quad \therefore \ \pi r^2 h=3V;$$

$$\therefore \ \mathbf{h}=\frac{\mathbf{3V}}{\pi \mathbf{r}^2}.$$

(ii) Further, $$r^2=\frac{3V}{\pi h};$$

$$\therefore \ \mathbf{r}=\sqrt{\left(\frac{\mathbf{3V}}{\pi \mathbf{h}}\right)}.$$

Example II. In an optical experiment, the following equation occurs:

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{zu}.$$

Make (i) u, (ii) z the subject of the relation.

(i) Multiply each side of the equation by $f\,.\,u$.

$$\therefore \ u=\mathbf{f}+\frac{\mathbf{f}}{\mathbf{z}}.$$

Note. This may also be written, $u=\dfrac{f(z+1)}{z}$.

(ii) Multiply each side of the equation by $f\,.\,u\,.\,z$

$$\therefore \ uz=fz+f;$$

$$\therefore \ zu-zf=f; \quad \therefore \ z(u-f)=f;$$

$$\therefore \ \mathbf{z}=\frac{\mathbf{f}}{\mathbf{u}-\mathbf{f}}.$$

Exercise XV. *a*

1. The following formula occurs in connection with simple interest. Can you interpret it ?

$$A = P + \frac{PRT}{100}.$$

Make (i) P, (ii) T the subject of the formula.
Then find T, if $A = 504$, $P = 420$, $R = 5$.

2. If a car increases its speed from u ft. per sec. to v ft. per sec. in t seconds, and if the rate of increase of velocity is a ft. per sec. per sec., then we have the formula $v = u + at$. Make a the subject of the formula.

3. If with the data of No. 2, the car travels s feet in this time, t seconds, we have the following formulae :

(i) $s = \frac{u+v}{2} \cdot t$; (ii) $s = ut + \frac{1}{2}at^2$;

(iii) $v^2 = u^2 + 2as$; (iv) $s = vt - \frac{1}{2}at^2$.

Make v the subject of (i). Make a the subject of (ii). Make u the subject of (iii). Make v the subject of (iv).

4. A path x ft. wide surrounds a lawn l yd. long, b yd. broad ; the total area of the path is A sq. ft. Express A in terms of x, l, b ; then make b the subject of the formula.

5. The time, t seconds, of a complete oscillation of a simple pendulum of length l ft. is given by the formula

$$t = 2\pi\sqrt{\left(\frac{l}{g}\right)}.$$

Make (i) l, (ii) g the subject. Then find g to two significant figures, if $t = 2$, $l = 3\frac{1}{4}$, $\pi = 3{\cdot}14$.

6. The area between two concentric circles of radii r inches and $(r + d)$ inches is A sq. in. Express A in terms of π, r, d ; then make r the subject.

7. The sum s of the first n of the numbers

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots$$

is given by the formula, $s = \frac{n}{2}\{2a + (n - 1)d\}$. Make (i) a, (ii) d, the subject.

8. In Fig. 245 a chord AB, of length l inches, cuts off from a circle of radius r inches a segment of height h inches. Prove that $l^2+4h^2=8rh$. Make (i) l, (ii) r, the subject. Then find r when $h=5$ and $l=12$.

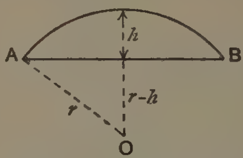

FIG. 245.

9. If an object is at a distance of u feet from a spherical mirror of focal length f feet, the distance of the image from the mirror is v feet where $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$. Make (i) u, (ii) f, the subject. Then find u, if $f=4$, $v=12$.

10. The proper distances of the photographic plate and of the person to be photographed from the lens of a camera, d, D inches, are connected by the formula, $Dd-a(D+d)=b^2$, where a, b are constant quantities, depending on the lens used. Make d the subject.

11. The capacity of condensers in series is given by the formula, $\frac{1}{c}=\frac{1}{c_1}+\frac{1}{c_2}$. Make c_1 the subject.

12. If V cu. in. of water are poured into a spherical bowl of radius r in., the depth of the water at the centre is h in., where $V=\pi h^2(r-\frac{1}{3}h)$. What does this become if $h=\frac{1}{2}r$? And in this case make r the subject.

13. Make w the subject of the formula, $E=\frac{wa}{(w+W)b}$.

14. Make h the subject of the formula, $a=b\cdot\frac{1+ch}{1-dh}$.

15. Make p the subject of the formula, $t=\frac{2pr}{p+2s}$.

16. Make t the subject of the formula, $s-a=\sqrt{\left\{\frac{ab}{t}+a^2\right\}}$

17. The least velocity for throwing a stone over a wall h feet high, c feet away, is v ft. per sec., where

$$(v^2-gh)^2=g^2(c^2+h^2) \quad \text{and} \quad g=32.$$

Express h in terms of g, c, v. Then find h when $c=36$, $v=48$.

18. If $\frac{a}{x}+\frac{b}{y}=\frac{a}{y}$. express (i) a in terms of b, x, y, (ii) x in terms of a, b, y.

19. If $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$ and $y=\frac{vx}{u}$, express x in terms of y, v, f.

Literal Equations.

It is generally understood, unless otherwise stated, or otherwise evident, that letters at the end of the alphabet—x, y, z, u, v, w—stand for unknown numbers, and other letters, a, b, c, etc. for known numbers.

Example III. Solve: $ax-ab=ab-bx$.

Here a, b are known numbers; we are asked to find x in terms of a, b.

Collect terms in x on one side, terms without x on the other side.

We have $$ax+bx=ab+ab;$$

$$\therefore \ x(a+b)=2ab.$$

Divide each side by $(a+b)$; $\quad \therefore \ x=\frac{2ab}{a+b}$.

Note. We use precisely similar methods and arguments in solving literal equations (*i.e.* equations containing letters) as we use in solving numerical equations.

Exercise XV. *b*.

Solve the following equations:

1. $4x-3a=5a+2x$. 2. $x+a=b$. 3. $ax=b$.

4. $a-x=x-b$. 5. $\frac{a}{x}=\frac{c}{b}$. 6. $b-x=c$.

7. $2(x+a)=5(x-a)$. 8. $abx=a+b$. 9. $a(x+b)=2ab$.

10. $3(x-p)=x+3q$. 11. $(m-n)x=mn$. 12. $ax+bx=c$.

13. $ax+b=bx+a$. 14. $x+ax=b$. 15. $ax-a^2=b^2$.

16. $\frac{a}{x}=2a.$ 17. $\frac{1}{x}=\frac{q}{p}.$ 18. $\frac{x}{p}=\frac{p}{q}.$

19. $\frac{x+a}{x-a}=2.$ 20. $\frac{a-x}{b-x}=3.$ 21. $\frac{cx}{a+b}=\frac{c}{d}.$

22. $\frac{1}{a}-\frac{1}{x}=\frac{1}{b}.$ 23. $\frac{x}{a}+\frac{x}{b}=1.$ 24. $\frac{a}{x}-1=1-\frac{b}{x}.$

25. $h(x-h)=k(x-k).$ 26. $a(x-b)=b(x-a).$

27. $\frac{x}{2}+\frac{a}{3}=\frac{2a}{3}-\frac{x}{3}.$ 28. $\frac{1}{2}(x+a)+\frac{1}{3}(x-a)=a.$

29. $\frac{a}{2}(x-b)=\frac{b}{3}(x-a).$ 30. $\frac{1}{a-b}-x=\frac{1}{a+b}.$

31. $ax+\frac{1}{a}=bx+\frac{1}{b}.$ 32. $\frac{x-a}{b}+\frac{x-b}{a}=2.$

33. $\frac{x-a}{b-a}=\frac{x-b}{a-b}.$ 34. $\frac{p-q}{qx+r}=\frac{p+q}{px-r}.$

35. $\frac{x-a}{x-b}=\frac{x-c}{x-d}.$ 36. $(x-a)^2-(x-b)^2=0.$

37. $x+y=a,\ x-y=b.$ 38. $x+ay=2,\ 2x-ay=1.$

39. $y=2x,\ y+mx=c.$ 40. $ax=by,\ x+y=c.$

Example IV. (i) Find the number which is as much above 5 as it is below 17.

(ii) Find the number which is as much above a as it is below b.

Let x be the unknown number.

In (i), we have
$$x-5=17-x;$$
$$\therefore\ 2x=17+5;$$
$$\therefore\ x=\frac{17+5}{2}=11.$$

In (ii), we have
$$x-a=b-x;$$
$$\therefore\ 2x=b+a;$$
$$\therefore\ x=\frac{b+a}{2}.$$

Note. If you do not see how to tackle a problem containing letters, invent a similar problem with numbers for letters; work it out and then use the same method for the given problem.

Exercise XV. c.

1. (i) A stick is 18 in. long It is cut into two parts, one of which is three times the other. Find the length of each part.

(ii) A stick is l in. long. It is cut into two parts, one of which is (1) three times the other, (2) n times the other. Find the length of each part.

2. (i) Find the angles of a triangle if each angle at the base is double the vertical angle.

(ii) Find the angles of a triangle if each angle at the base is n times the vertical angle.

3. The sum of two consecutive integers is $2n+15$; find them.

4. On one side of a pair of scales is a weight of W lb.; on the other side is a smaller weight of w lb. How much must be taken from the first side, and put on the other side, so that they may balance?

5. One angle of a triangle is $d°$, the other two are equal. How much is each of them?

6. The perimeter of a rectangle is p in.; its length is l in.; what is its breadth?

7. Six ordinary chairs and two arm-chairs cost £a; each ordinary chair costs p shillings less than an arm-chair; find the cost in shillings of an arm-chair.

8. The same number is added to the numerator and denominator of the fraction $\frac{a}{b}$, and its value is then $\frac{1}{2}$; what number is added?

9. A room is a ft. long, b ft. wide; the area of the walls is $2ab$ sq. ft. Find the height.

10. In a two-bladed pocket-knife of ordinary pattern, the distance between the tips of the two blades is b cm. when both are closed and is c cm. when both are fully open. Find the distance between the pins about which the blades pivot.

11. A chain hangs over a small peg; l ft. of it are on the longer side and s ft. on the shorter side; what is the distance of the mid-point of the chain from the peg? (See Fig. 249).

12. The price of coal diminishes r per cent. and is then c shillings per ton. What was it at first?

13. Divide A shillings between two boys, so that one has m shillings more than the other.

14. Divide p into two parts in the ratio a to b.

15. If r per cent. of a man's income is £p, what is his income?

16. Pewter consists of t parts tin to l parts lead; how much tin is there in p lb. of pewter? What is the answer if $t=4$ and $l=1$?

17. If, in Fig. 246, AD bisects $\angle BAC$, it can be proved that

$$\frac{BD}{DC}=\frac{BA}{AC}.$$

(i) If $BC=7$ in., $CA=4$ in., $AB=6$ in., find BD, DC.
(ii) If $BC=a$ in., $CA=b$ in., $AB=c$ in., find BD, DC.

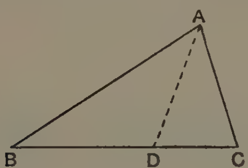

FIG. 246.

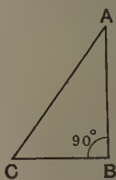

FIG. 247.

18. In Fig. 247, $AB=(p^2-q^2)$ in. and $AC=(p^2+q^2)$ in., find BC.

19. In Fig. 248, $AO=OP=OB$ and $AN=p$ in., $NB=q$ in., find OP and PN.

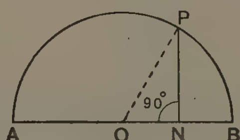

FIG. 248.

EASY REVISION PAPERS. A. 36-45

A. 36

1. If $v^2 = u^2 + 2as$ and if $v = u + at$, find s in terms of u, v, t only. Then find s if $u = 3$, $v = 7$, $t = 6$.

2. (i) Divide $x(a+b) + y(a+b) - z(b+a)$ by $a+b$.

(ii) Can you factorise $ax + x^2 - a + x$? If so, do so and test your answer by multiplication.

3. (i) Solve $$\frac{5x-9}{2x-1} - 2 = \frac{2x-5}{2x-1}.$$

(ii) If $12x + 7y = 29$ and $6x - 5y = 23$, find the value of $2(x+y)$.

4. Simplify $$\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} - \frac{2}{x^2-y^2}.$$

5. The effect of quadrupling the duty on wine is to increase the price of a bottle of wine by 50 per cent. ; the increase of cost is the same as the increase of duty. Find what percentage of the original price was duty.

A. 37.

1. (i) For what values of x is $\dfrac{4x-4}{6x-6}$ equal to $\dfrac{2}{3}$?

(ii) Simplify $$1 - \frac{3x}{4x-4} - \frac{5}{6x-6}.$$

2. If $v = at$ and $s = \frac{1}{2}at^2$ express s in terms of

(i) v, a only ; (ii) v, t only.

3. Draw the graph of $x^2 - x$ for values of x from -2 to 3. Use it to solve $x^2 - x = 5$. For what values of x is x^2 equal to x ?

4. Solve (i) $ax = 1$;

(ii) $2x^2 - x - 4 = 0$ (to one place of decimals).

FIG. 249.

5. Fig. 249 shows a chain ACB hanging over a small rail. Find AC and BC, (i) if the chain is 9 ft. long and A is 18 in. above B, (ii) if the chain is l ft. long and if B is s ft. below A.

A. 38

1. Write down the values of

(i) $-x^2 - (-2x^2)$; (ii) $(-2x) \div (-3x)$.

Find the number half-way between $a-b$ and $b-a$.

2. Factorise (i) $x(2x+5)+3$; (ii) $x(x+7)-6(x+2)$. Using "the difference of two squares," write down the value of 105×95.

3. Solve (i) $\dfrac{3x-5}{2x+1}=\dfrac{6x-11}{4x+3}$; (ii) $\dfrac{x}{2}+y=x+\dfrac{y}{2}=2$.

4. State three special cases of the following theorem and verify them. If x is any whole number, the last digit of x^5 is the same as the last digit of x.

5. At a shooting range, one pays twopence for a miss and receives sixpence for a hit. A man has 27 shots and has to pay 1s. 10d.; how many hits did he make?

A. 39

1. Simplify (i) $(x^2+12x+35)(x^2-x-20)\div(x^2-25)$;

$$\text{(ii)}\ \left(x+12+\frac{35}{x}\right)\left(x-1-\frac{20}{x}\right)\div\left(x-\frac{25}{x}\right).$$

2. Find the remainder when x^3+7x^2-15x is divided by $x-2$. What number added to x^3+7x^2-15x will make the result divisible by $x-2$?

3. From the optical formula $\dfrac{1}{x_1}+\dfrac{1}{x_2}=\dfrac{2}{r}$, find x_1 in terms of x_2 and r. Then *write down* x_2 in terms of x_1 and r.

4. Solve (i) $\dfrac{x}{2}-\dfrac{x-1}{3}=17$; (ii) $\dfrac{x}{a}+b=\dfrac{x}{b}+a$.

5. Draw the graph of $4x-3x^2$ from $x=-1$ to $x=4$. Use it (i) to solve $4x-3x^2=1$; (ii) to solve $3x^2-4x+1=0$; (iii) to find one root of $4x-3x^2+21=0$. Can you then say what the other should be?

A. 40

1. Simplify $\dfrac{x^4-1}{(x^2+1)^2}$. If the result is equal to $1+\dfrac{a}{x^2+1}$, find the value of a.

2. AC is the diameter of the semicircle ABC. It is 50 yards shorter to go from A to C along AC than along ABC; find the length of AC to the nearest yard.

3. Find the factors of (i) $px-py+qy-qx$;

$$\text{(ii)}\ (x^2-6x+3)^2-(x-9)^2.$$

4. Solve (i) $x+2y=2x-3y+7=0$;

(ii) $\frac{a}{x}+\frac{b}{x}=\frac{2}{a}+\frac{2}{b}$.

5. A man used to spend 25s. a week on rent and food. His rent has now increased 10 per cent. and the cost of his food 20 per cent., so that he now spends 29s. a week. Find his original rent.

A. 41

1. (i) Multiply x^2-xy+y^2 by $x+y$.

(ii) Simplify $(x^3+y^3)\div(x^2-xy+y^2)$.

(iii) Simplify $\frac{x+y}{x^2-xy+y^2}+\frac{2}{x+y}$.

2. Find t in terms of p, v if $\frac{pv}{273+t}=10$.

3. Solve (i) $\frac{x+1}{\frac{1}{2}}-\frac{x-1}{\frac{1}{3}}=\frac{x-3}{\frac{1}{4}}$;

(ii) $x^2+2y^2=17,\ 3x^2-5y^2=7$.

4. Take any number of two digits and the number formed by reversing the digits and show that the difference of these numbers is divisible by 9. Prove this is always true.

5. A man buys 1000 bulbs for 19s., some at the rate of 50 for 9d. and the rest at 100 for 2s. 6d. How many did he buy of each kind ?

A. 42

1. If $a+b=5$ and $x-y=1$, write down, if possible, the numerical values of

(i) $b+a$; (ii) $y+x$; (iii) $\frac{b+a}{y-x}$; (iv) x^2-y^2; (v) $a^2+2ab+b^2$.

2. Simplify (i) $\left(\frac{a}{b}-\frac{b}{a}\right)\div\left(\frac{1}{a}+\frac{1}{b}\right)$;

(ii) $\frac{y-z}{yz}+\frac{z-x}{zx}+\frac{x-y}{xy}$.

3. Solve (i) $1-\frac{x}{x+2}=\frac{x+2}{x+1}-1$;

(ii) $2xy-7x=7,\ 3xy-10x=14$.

4. Simplify $(y+z)(y-z)+(z+x)(z-x)+(x+y)(x-y)$.

5. I spend £1 in laying in a stock of sugar and another £1 when the price has gone up 1d. per lb. Altogether I bought 108 lb. What were the prices ?

A. 43

1. An important formula in Dynamics is $P=\dfrac{Wv^2}{gr}$; find (i) r in terms of P, g, W, v ; (ii) v in terms of P, g, r, W.

2. (i) Find the H.C.F. of $2ax+x^2$ and $x(x^2-4a^2)$.

(ii) Simplify $\quad \dfrac{a}{2ax+x^2}-\dfrac{a}{2ax-x^2}-\dfrac{2a}{x^2-4a^2}$.

3. Factorise $\quad (ax+by)^2+(bx-ay)^2$.

4. Solve (i) $6x^2=x+3$, give each root to two significant figures

(ii) $x+y=2a,\ ax-by=a^2+b^2$.

5. A flat ring is bounded by two concentric circles. The radius of the inner circle is r in. and the breadth of the ring is h in. ; the area of the ring is A sq. in. ; express h in terms of π, r, A.

A. 44

1. (i) Find the factors of $x(x+z)-y(y+z)$.

(ii) Simplify $\left(\dfrac{x}{y+z}-\dfrac{y}{x+z}\right)\cdot\dfrac{(y+z)+(x+z)}{x+y+z}$.

2. If $\dfrac{1}{u}-\dfrac{1}{v}=\dfrac{1}{k}$, find v in terms of u, k.

3. Solve (i) $\dfrac{2x-2}{y-5}=\dfrac{x-4}{2y-6}=2$; (ii) $\dfrac{x-a}{b}=\dfrac{x+b}{a}-2$.

4. 1 dollar + 1 franc = 5 shillings ; 1 franc + 1 shilling = 0·44 dollars ; 1 shilling + 1 dollar = 6·2 francs. Find the value of a dollar and a franc in English money. Are all these equations necessary ? Do they contradict one another ?

5. Soldiers march at 3·6 miles an hour for 50 minutes, then halt for 10 minutes, then march again at the same rate for 50 minutes, and so on. Draw a graph to show their position after any time up to 5 hours. A cyclist gives them a start of $1\frac{2}{3}$ hours and then follows them at 12 miles an hour. When does he catch them up ? Are they walking or halted, when overtaken ? Solve graphically.

A. 45

1. If you write down the whole numbers beginning with m and going up as far as n, inclusive, how many will there be ?

2. Solve (i) $(x+1)(x-2)-(x-3)(x+4)=(x-2)(x-3)$;

(ii) $x=y+6,\ x^2=y+48$.

3. AB is 12 in. long. C is a point on AB, such that

$$AC \,.\, CB = 35 \text{ sq. in.}$$

Find the length of AC.

4. A motor cycle is bought for £45 and sold at a gain of x per cent. Express the selling price in terms of x. If the selling price is £40 10s., write down an equation for x and solve it. Interpret the answer.

5. (i) To convert degrees of Fahrenheit to Centigrade, one rule is "Subtract 32 and multiply the result by $\frac{5}{9}$"; another rule is "Add b, multiply the result by $\frac{5}{9}$, subtract b." For what numerical value of b do these rules agree ?

(ii) To convert degrees Centigrade to Fahrenheit, one rule is "Add c, multiply the result by $\frac{9}{5}$, subtract c." Use either rule in (i) to find the numerical value of c.

HARDER REVISION PAPERS. B. 21-30

B. 21

1. If the diameter of a sphere is d inches, its volume in cu. inches exceeds $\frac{1}{2}d^3$ by about 5 per cent. Find an approximate expression for the volume of a sphere of radius r inches.

2. Factorise (i) $12a^2+3ad-8ac-2cd$;

(ii) $x(x+1)-3y(3y+1)$.

3. Simplify $$1+\frac{3}{x+1}+\frac{2}{(x+1)^2}.$$

4. Solve (i) $2x-(a-x+a+x)=2b$;

(ii) $p+\dfrac{x}{q}=q+\dfrac{x}{p}$.

5. From Bristol to London is 120 miles ; an express train starts 6 minutes late from Bristol, but by travelling 2 miles an hour faster than its ordinary rate arrives punctually in London. Find its ordinary rate.

B. 22

1. Solve (i) $2x-1=\frac{1}{2}(x-\frac{1}{3})$; (ii) $2(x-1)=\frac{1}{2}(x+\frac{1}{3})$.

2. Simplify $$\frac{(a+b)x}{(x+a)(x-b)}-\frac{(b+c)x}{(x+c)(x-b)}.$$

3. Find the values of x and y correct to one place of decimals, given that $x^2+y=2\cdot27$ and $x=y+1\cdot3$.

4. A man arrives at Charing Cross and wants to catch a train at Paddington; a four-wheeler at 8 miles an hour will make him 4 minutes late, a bus at 10 miles an hour will give him 2 minutes to spare; how long has he for the journey?

5. Two thermometers A and B are both graduated; on A the freezing point is marked 32° and the boiling point 212°, but neither of these points is indicated on B. The following pairs of corresponding readings are observed: A 59°, B 35°; A 77°, B 45°. Draw a graph to show the readings on B which correspond to the A readings from $-25°$ to 212°. Use the graph to find (i) the freezing point and boiling point on B; (ii) the readings on each thermometer corresponding to zero on the other; (iii) the temperature for which the readings of A and B are the same.

B. 23

1. Find, to one place of decimals, the values of x, y, given that
$$\frac{x}{1\cdot2}+0\cdot4y=5 \quad \text{and} \quad \frac{x}{2\cdot7}-0\cdot3y=0\cdot75.$$

2. (i) Simplify $$\left\{a-\frac{b(2a-b)}{a+b}\right\}\div\left\{\frac{a^2}{a-b}-b\right\};$$

(ii) Factorise $(a+3)(a-1)-(2a-3)(a+1)$.

3. Solve $$\frac{3}{x-2}-\frac{x+1}{4-x}=1.$$

4. The resistance R lb. per ton to the motion of a train is given by the formula $R=a+bv^2$ where v is the speed in miles per hour and a, b are the same for all values of R, v. When $v=20$, $R=9\cdot6$, and when $v=40$, $R=20\cdot4$. Find the values of a and b, and find R when $v=35$, correct to 2 significant figures.

5. Show that the sum of any four consecutive integers is equal to the difference between the product of the two greatest and the two least.

B. 24

1. Find a formula for converting a price in farthings per lb. to the price in shillings per cwt. [If x shillings per cwt. is the same rate as f farthings per lb., express x in terms of f.]

2. (i) Add $\frac{a^2+1}{a^2-1}$ and $\frac{a^2-1}{a^2+1}$;

(ii) Express $\frac{x-2}{x^2-1}$ in terms of y, if $xy=1$.

3. Solve $\frac{x}{y}+3=7y, \quad (y-1)^2-x=0.$

4. A number of two digits is $5\frac{1}{2}$ times the sum of its digits; prove that the digits must be equal.

5. A bankrupt can just pay 7s. 6d. in the £; if his debts had been £1200 more, he could only have paid 6s. 8d. in the £. What are his debts ?

B. 25

1. Eggs at x a shilling cost the same as at yd. per dozen. Find an equation between x and y.

2. Find the H.C.F. of $3x^4+3x^3-6x^2$ and $4x^3-2x^2(x^2+1)$.

3. If $y=\frac{ax+1}{x-a}$, express x in terms of y, a. Find the value of a to two places of decimals if $y=2{\cdot}6$ when $x=3{\cdot}2$.

4. Solve $x=\frac{x-1}{x-3}$ (correct to two places of decimals).

5. The straight line AB is bisected at O and produced to any point P; prove that $OP^2=OA^2+AP\,.\,BP$.

B. 26

1. Multiply $3-\frac{1}{x+1}$ by x^2+x.

2. Factorise

(i) $x^4-7x^2y-18y^2$; (ii) $2ab-a^2-b^2$; (iii) $(x^2+1)^2-4x^2$.

3. What values of x (correct to one decimal place) make $2x^2+3$ equal to $8x$.

4. A hollow cube is made out of a rectangular piece of thin card-board, $6x$ in. long, $4x$ in. wide; what is its volume ?

5. Gold weighs 1200 lb. per cu. ft.; copper weighs 550 lb. per cu. ft. What weight of copper must be added to 1 lb. of gold, so that the alloy may weigh 1130 lb. per cu. ft. ?

B. 27

1. The following formula is given for the speed of a paddle-steamer: If P = indicated H.P., S = mid-sectional area in sq. ft., V = velocity in knots, then $V=\sqrt[3]{\left\{\dfrac{620P}{S}\right\}}$. Make P the subject of the formula. In what ratio must P be altered if V is to be doubled?

2 Prove that
$$(x-1)(2x+3)+(2x+1)(x-5)\equiv(3x+2)(x-3)+(x-2)(x+1).$$

3. (i) What number must be added to x^2-5x-3 to make the result a perfect square.

(ii) Express x^2-6x-7 as the difference of two squares.

4. Solve (i) $x+\dfrac{x}{a}=a$; (ii) $\dfrac{2}{x}-\dfrac{x}{2}=\dfrac{15}{4}$.

5. If $V=\pi r^2 h$ and $S=2\pi r^2+2\pi rh$, express V in terms of π, r, only. Interpret the formulae.

B. 28

1. Prove that 1, 3, $-\frac{1}{2}$ are the solutions of the equation
$$2x(x^2+1)-7(x^2-1)=4.$$

2. By what must $\dfrac{1}{x-1}$ be multiplied to give $\dfrac{1}{x}-1$?

3. If $V=\frac{4}{3}\pi r^3$ and $S=4\pi r^2$, express V in terms of π, S only. Interpret the formulae.

4. The sum of one pair of angles of a triangle is 105°, and the difference of another pair is 45°. Prove that the triangle is isosceles.

5. A train travels k miles from A to C and passes a station B on the way. The whole journey takes p hours. Its average speed from A to B is s miles an hour and from B to C is t miles an hour. How far is B from A?

B. 29

1. (i) If $\dfrac{1}{x}-\dfrac{2}{y}=1$, prove that $\dfrac{2}{x+1}-\dfrac{1}{y+1}=1$.

(ii) Simplify $\left(1+\dfrac{2}{x-1}\right)\times\left(x-\dfrac{2x}{x+1}\right)$.

2. If $A=\pi r^2$ and $C=2\pi r$, express A in terms of C, π only. Interpret the formulae.

3. Solve $$(x+5)^2-4(x+5)-12=0.$$

4. A gives B (x^2-y^2) miles start; A travels x miles an hour, B travels y miles an hour; how long will A take to catch up B, and how far from the starting point will he be ?

5. The ratio of the number of the sides of two convex polygons is 2 : 3, and the ratio of the sums of their angles is 3 : 5. Find the number of sides of each. [The sum of the angles of a convex polygon with n sides is $(2n-4)$ right angles.]

B. 30

1. Two telegraph poles (see Fig. 250) of equal height are at distance $2l$ feet apart, and the connecting wire is of length s feet.

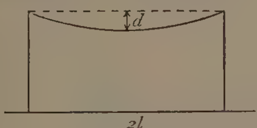

FIG. 250.

The sag at the mid-point is d feet, where $d=\frac{1}{2}\sqrt{(3ls-6l^2)}$. (i) Make s the subject of this formula. (ii) If $x=s-2l$, prove that $x=\frac{4d^2}{3l}$. (iii) If the poles are 30 yards apart and if the greatest sag is 6 inches, by how much will the length of the wire exceed the distance between the poles ?

2. The sides of a triangle are $2p+1$, $2p(p+1)$, $2p^2+2p+1$ inches; prove that the triangle is right-angled. What special case is given by $p=3$?

3. Solve the equation $2x+5=\frac{1}{2x+5}$.

4. A line AB, 5 inches long, is produced to a point C such that $AC \cdot CB=6AB^2$. Calculate the length of BC.

5. In a mill, the men earn p shillings a day and the women earn q shillings a day; there are n people employed and the average wage is A shillings per day. How many men are employed ?

SUPPLEMENTARY EXERCISE. S. 11

Miscellaneous Problems.

1. In Fig. 251, both arrows point East. Calculate x.

2. If n cells, each of internal resistance r ohms, are arranged in series, and if the external resistance is R ohms, and if the E.M.F. is E volts, the current C amperes is given by

$$C=\frac{nE}{R+nr}.$$

Make n the subject of the formula.

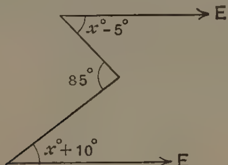

Fig. 251.

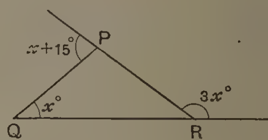

Fig. 252.

3. In Fig. 252, calculate $\angle PQR$.

4. How many measurements must be taken in order to make a copy of (i) a triangle, (ii) a quadrilateral, (iii) a pentagon (5 sides), (iv) an octagon (8 sides). If M measurements are required for copying an n-sided polygon, find a formula for M in terms of n.

5. In Fig. 253, the arrows point respectively due East and due West. Calculate x.

6. What digit must x represent if the product of the two numbers "$x1$" and "$4x$" is the number "$3xxx$"?

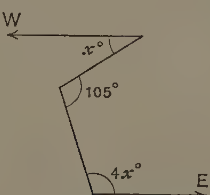

Fig. 253.

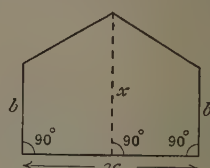

Fig. 254.

7. Fig. 254 represents the end wall of a house, measurements in feet; the area is A sq. ft.; express A in terms of x, b, c, and then make x the subject of the formula.

8. For what integral values of x does $\dfrac{x+3}{2x+1}$ lie between $\frac{2}{3}$ and $\frac{3}{4}$?

9. If, in Fig. 255, PN is parallel to AB, then $\dfrac{PN}{NC}=\dfrac{AB}{BC}$. If $AB=7$ in., $BC=3$ in., find CN, so that PN equals NB.

10. A cyclist rides 7 miles per hour uphill and 14 miles per hour downhill. His average speed on the hills is 12 miles per hour. What is the ratio of the downhill to the uphill portion?

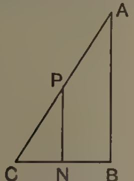

FIG. 255.

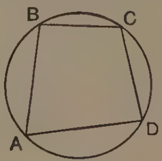

FIG. 256.

11. In Fig. 256, if $\angle BAD=\frac{3}{5}\angle BCD$, calculate $\angle BAD$. [The opposite angles of a cyclic quadrilateral add up to two right angles.]

12. The sides of a box are x, $x-1$, $2x$ inches long and its diagonal is $(2x+1)$ inches long. Find the value of x.

13. There is a misprint in one (but only one) of the coefficients of the following set of equations: $15x+y-4z=13$; $7x-11y+13z=76$; $3x+8y+5z=55$. The correct answer is $x=2$, $y=3$, $z=5$. Which equation is wrong? Find all the possible corrections.

14. In Fig. 257, ABC is a section of a spherical mirror, centre O, radius 40 in.; the image of a small object at P is at

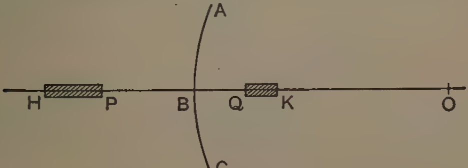

FIG. 257.

Q. If $BP=y$ in., $BQ=z$ in., then $\dfrac{1}{z}-\dfrac{1}{y}=\dfrac{1}{20}$ for all positions

of P on OB produced. (i) Find the length of OQ, if $BP=5$ in. (ii) If PH is a thin rod, 2 in. long, what is the length of its image QK? (iii) How far from O is the image of a very distant object on OB produced?

15. Is it possible to find three consecutive integers a, b, c, such that $a^2+c^2=2b^2$?

16. If $x=1\frac{1}{2}$ is one root of $2x^2+x+c=0$, what is the other root? Then find the value of c.

17. Fig. 258 represents a section of a bolt formed of two circular cylinders of diameters p in., q in.; what is the weight of the bolt if it is solid, and if the metal weighs W lb. per cu. in.?

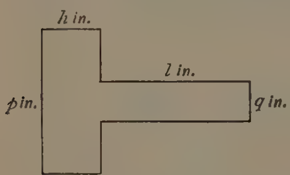

FIG. 258.

18. (i) If n is an even integer, what is the coefficient of x^n in the series $1+3x^2+5x^4+7x^6+9x^8+\ldots$?

(ii) If n is an odd integer, what is the coefficient of x^n in the series $2x+3x^3+4x^5+5x^7+6x^9+\ldots$?

19. The perimeter of a triangle is 4 feet, and one side is twice another side. Prove that the length of the shortest side lies between 8 inches and 1 foot.

20. Use the fact mentioned in No. 11 to calculate the value of x in Fig. 259.

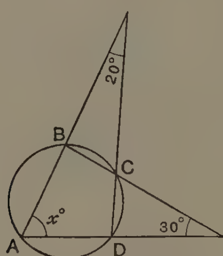

FIG. 259.

21. Prove without solving the equation that one root of $7x^2-45x-11=0$ is greater than 6.

22. Write down any sum of money £x ys. zd. where $12>x>z$; subtract from it the reversed sum £z ys. xd. Suppose the result is £p qs. rd.; add to this the reversed sum £r qs. pd. Prove that the result is £12 18s. 11d.

23. If $x+3y+7z=14$ and $x+4y+10z=17$, show that there is a numerical value of c for which it is possible to find the *numerical* value of $x+5y+cz$ from the data. What is this value of c?

24. In Fig. 260, calculate the lengths of AQ and BP. [The tangents from a point to a circle are equal, $AQ=AR$, etc.]

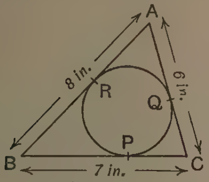

FIG. 260.

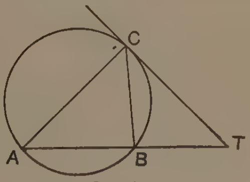

FIG. 261.

25. In Fig. 261, it can be proved that

$$\frac{BC}{CA}=\frac{TB}{TC}=\frac{TC}{TA}.$$

(i) If $BC=3$ in., $TB=6$ in., $TC=8$ in., find the lengths of AC and AB.

(ii) If $BC=5$ in., $CA=6$ in., $AB=8$ in., find TB and TC.

(iii) If $BC=6$ in., $CA=5$ in., $AB=8$ in., find TB and interpret the answer.

26. From the data of a problem in which y is the number of articles bought and £x is their total cost, a boy obtains the following equation:

$$\frac{1{\cdot}5+0{\cdot}3(y-x)}{y+x}+0{\cdot}3=\frac{1{\cdot}5}{x}.$$

Is it possible for him to find the *numerical* value of x, without obtaining a second equation?

27. With the data of Fig. 262, express x in terms of a, b.

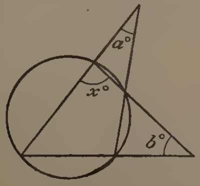

FIG. 262.

28. A common type of error when using an automatic adding machine is to record sums such as £7 5s. as 7s. 5d., or £13 9s. as 13s. 9d. If only one amount is recorded incorrectly in this manner and if the error in the result is £x ys. zd., prove that $x+y+z$ is either 11 or 30, and that the wrong entry is either xs. $(12-z)$d. or $(x+1)$s. $(12-z)$d.

29. Find all the integral values of x which make $(x-3)^2$ lie between $3x-5$ and $13-3x$.

30. The lengths of two altitudes of a triangle are 3 in. and 5 in.; prove that the length of the third altitude must be less than $7\frac{1}{2}$ in. What length must the third altitude exceed? [The area of a triangle equals $\frac{1}{2}$ height $\times$ base; any two sides of a triangle are together greater than the third side.]